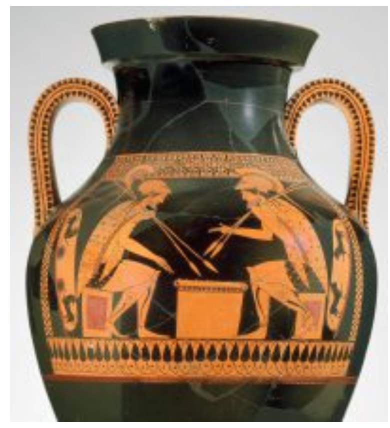
Best responses in repeated games
Reactive strategies with longer memory.

LEG March 2025



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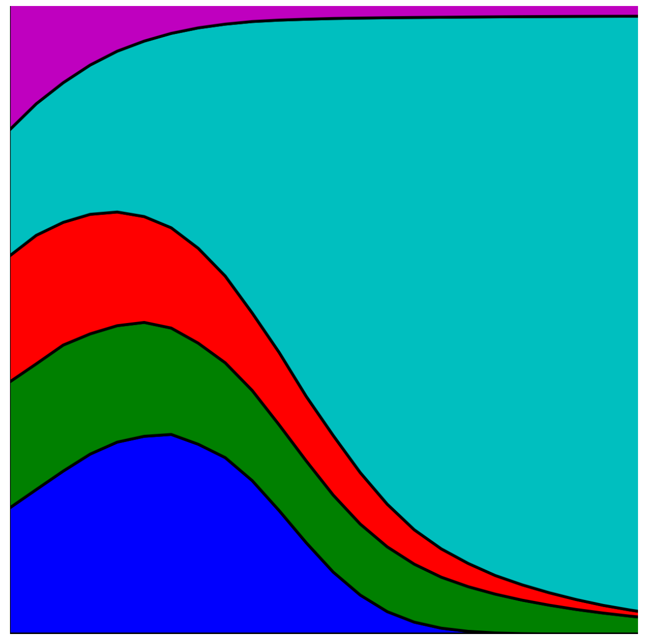
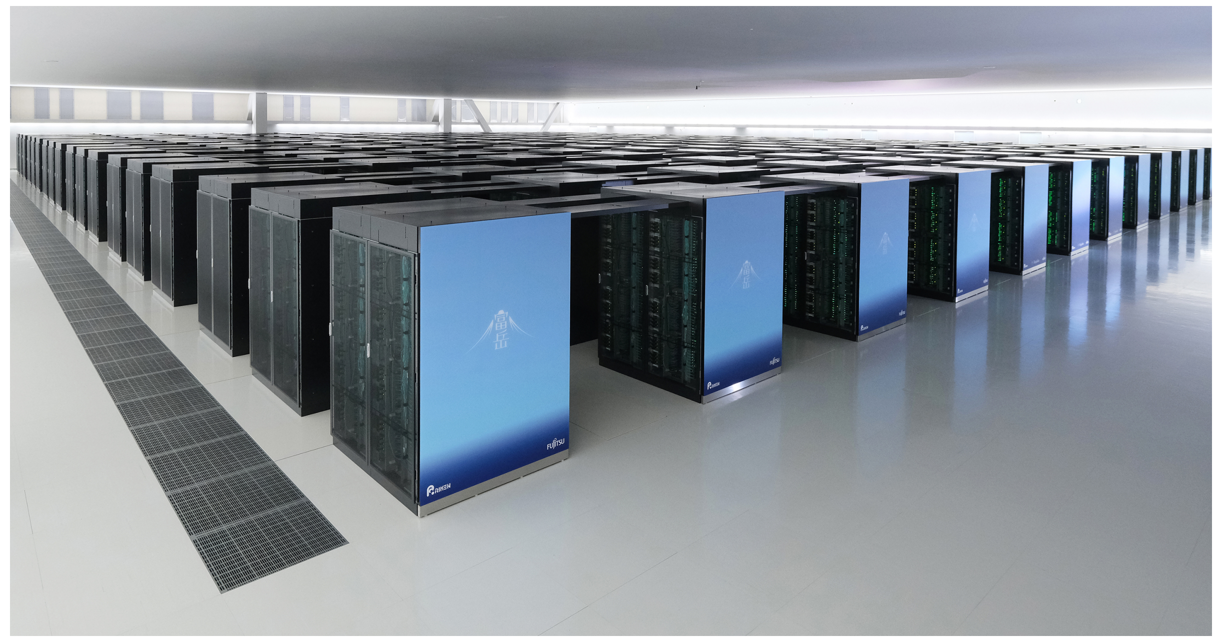
Nikoleta Glynatsi

MAX-PLANCK-GESELLSCHAFT

RIKEN

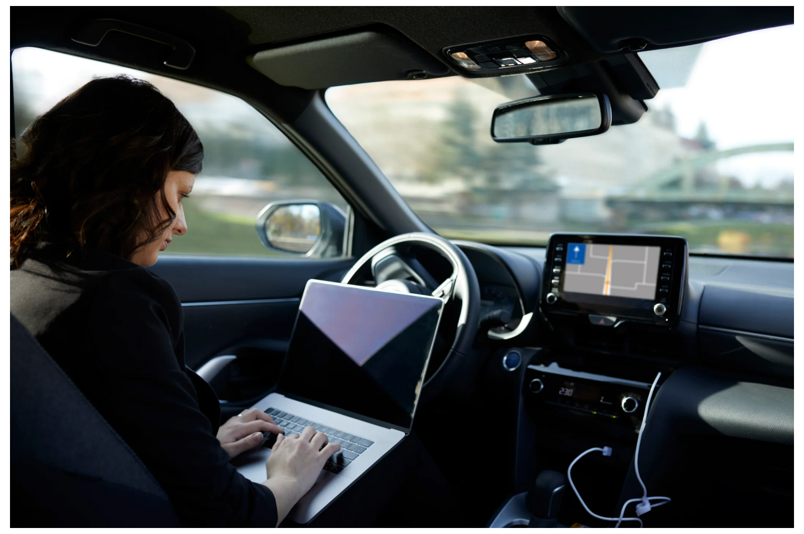
R-CCS





Social Behavior

Understand Cooperation



1. Introduction and motivation

2. Conditional cooperation with longer memory

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3.

Complete strategy spaces of direct reciprocity

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Can I afford to remember less than you?

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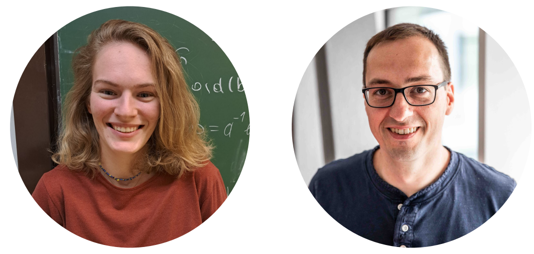

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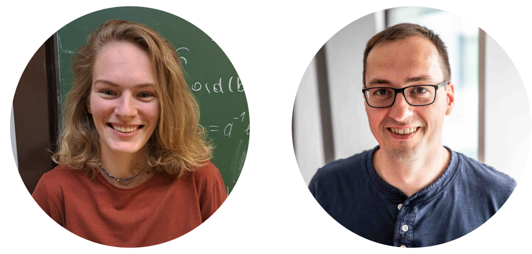

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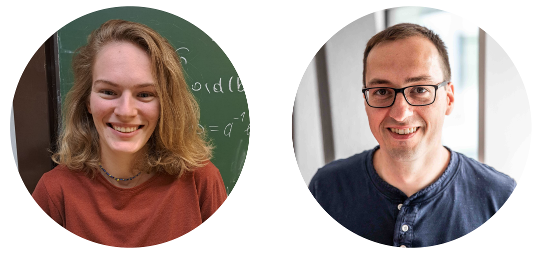

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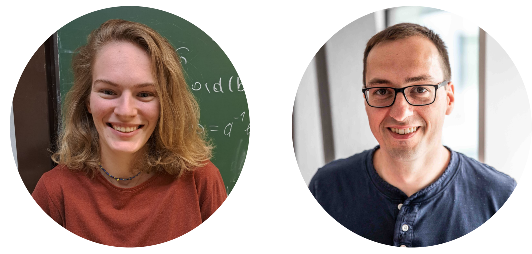

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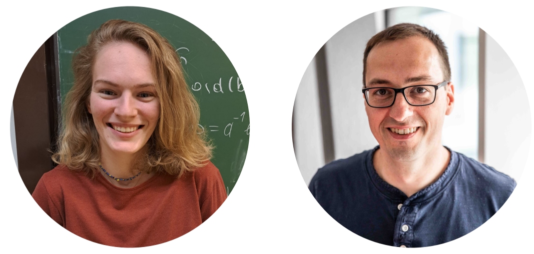

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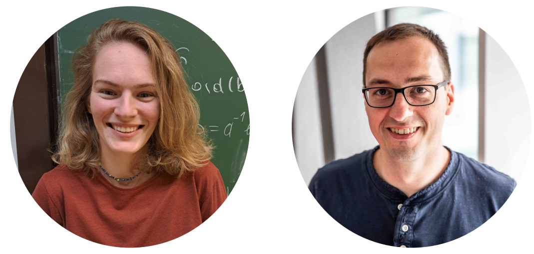

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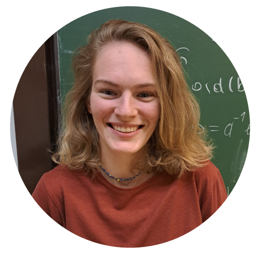
Economics Letters  

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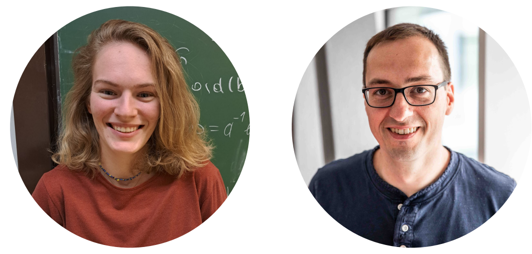

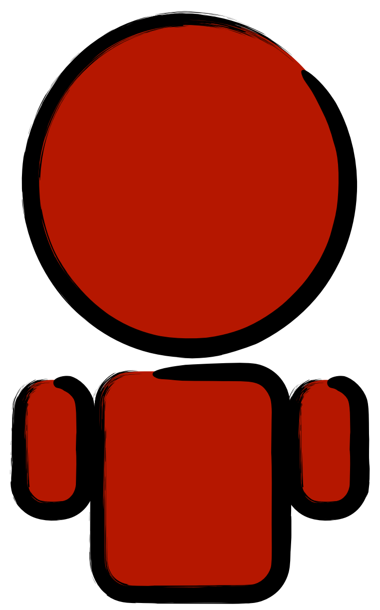

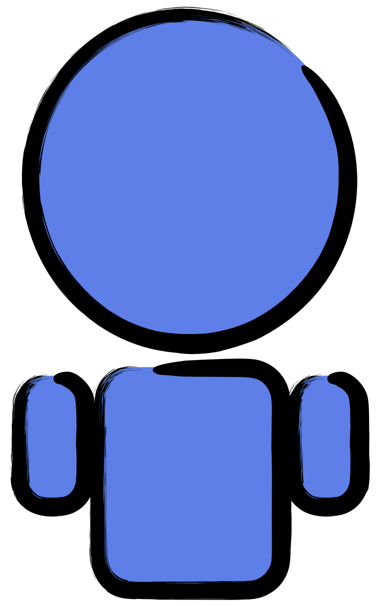

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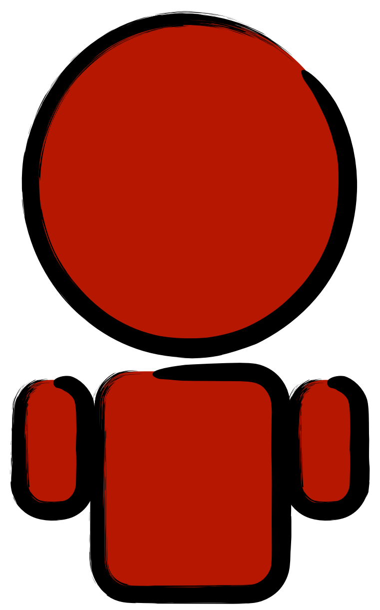

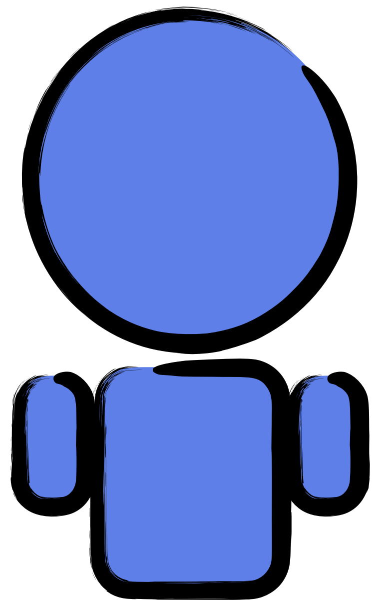

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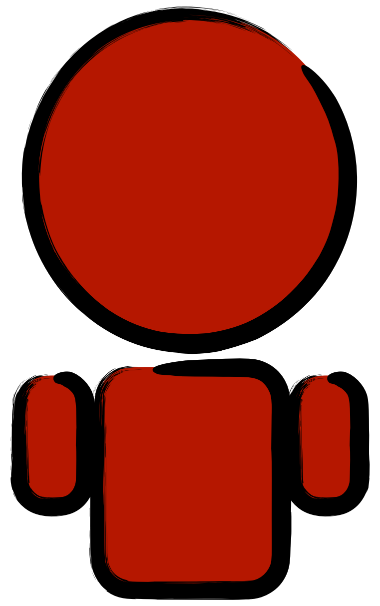

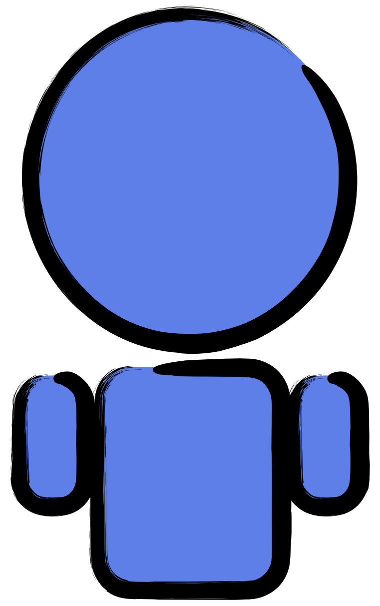

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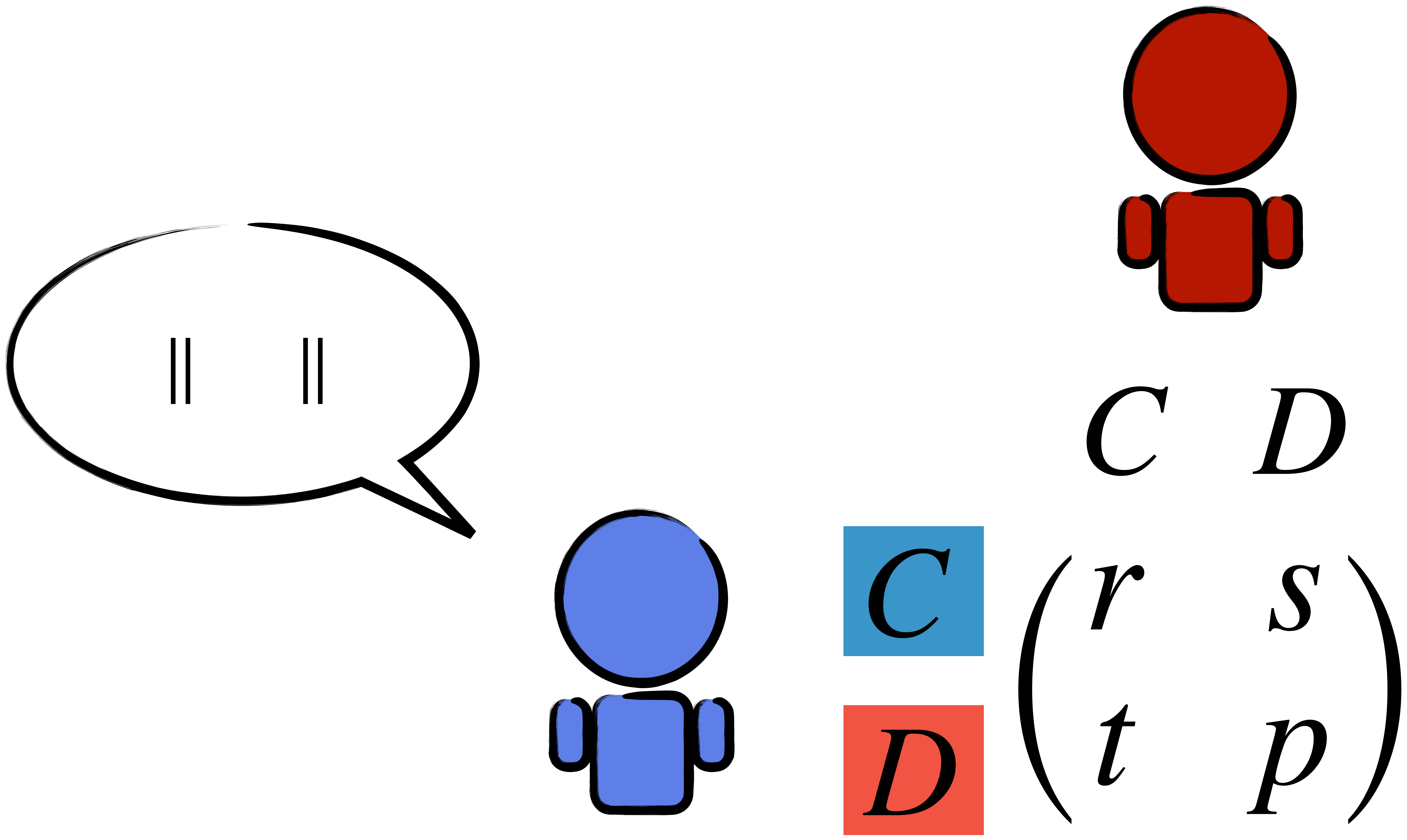
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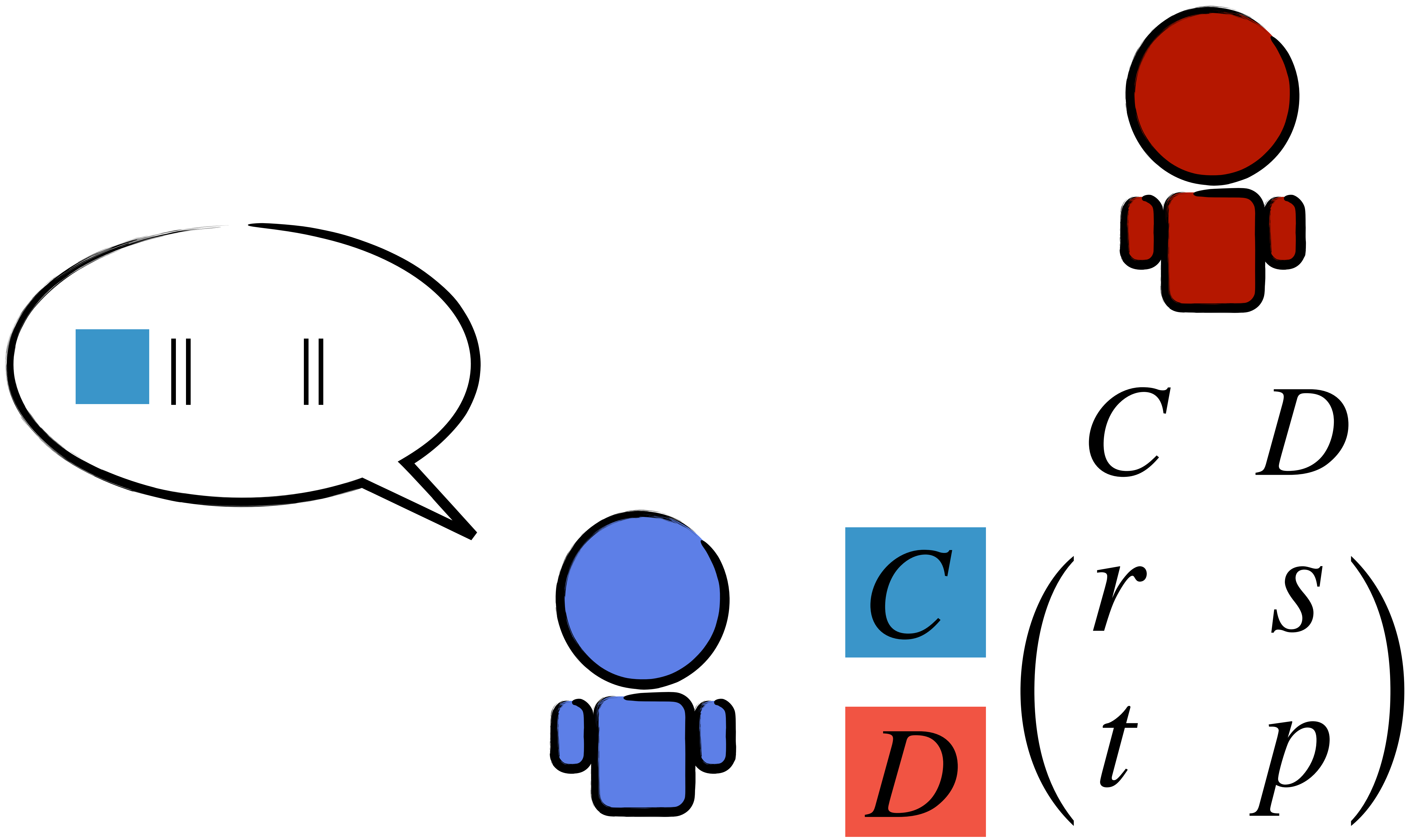

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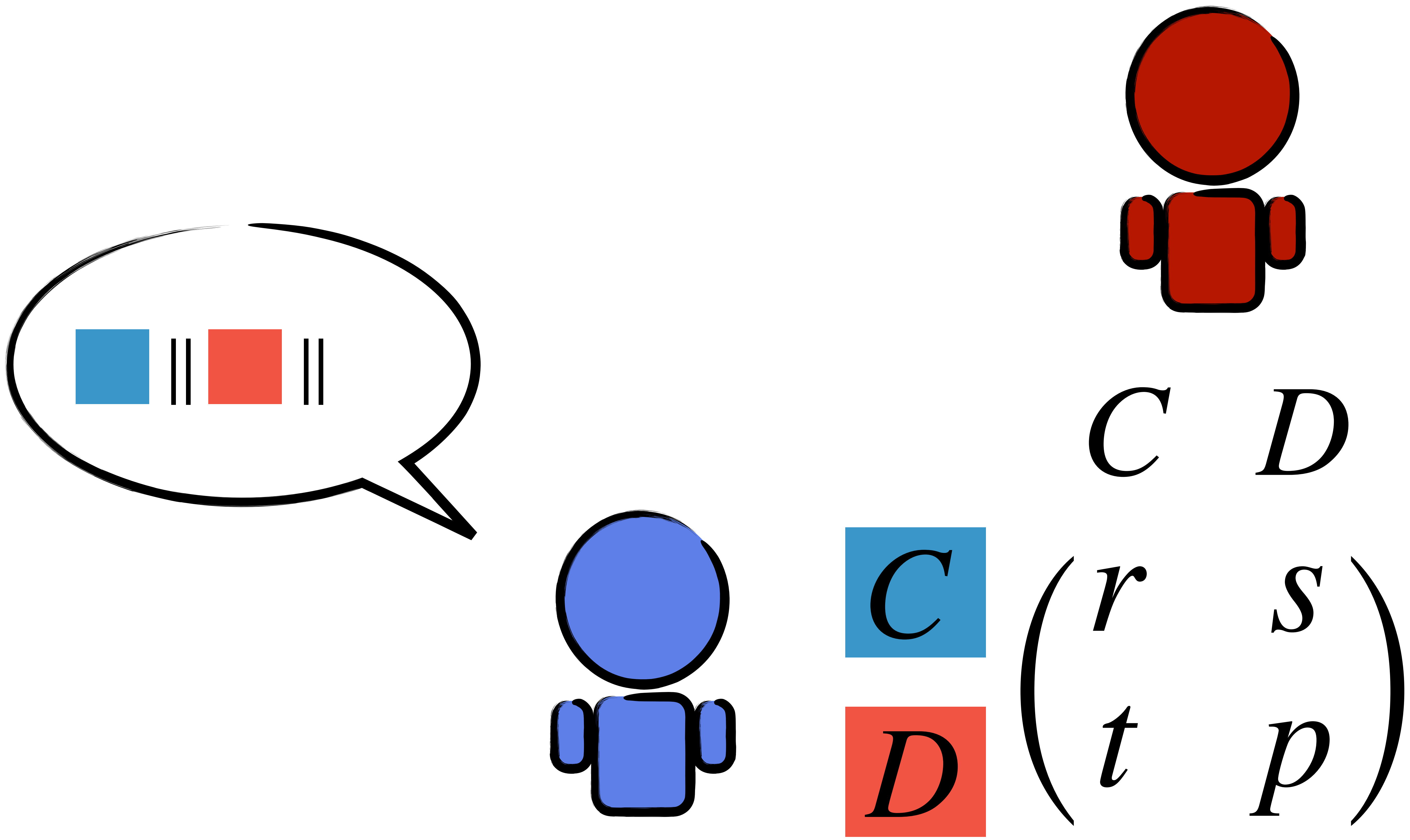

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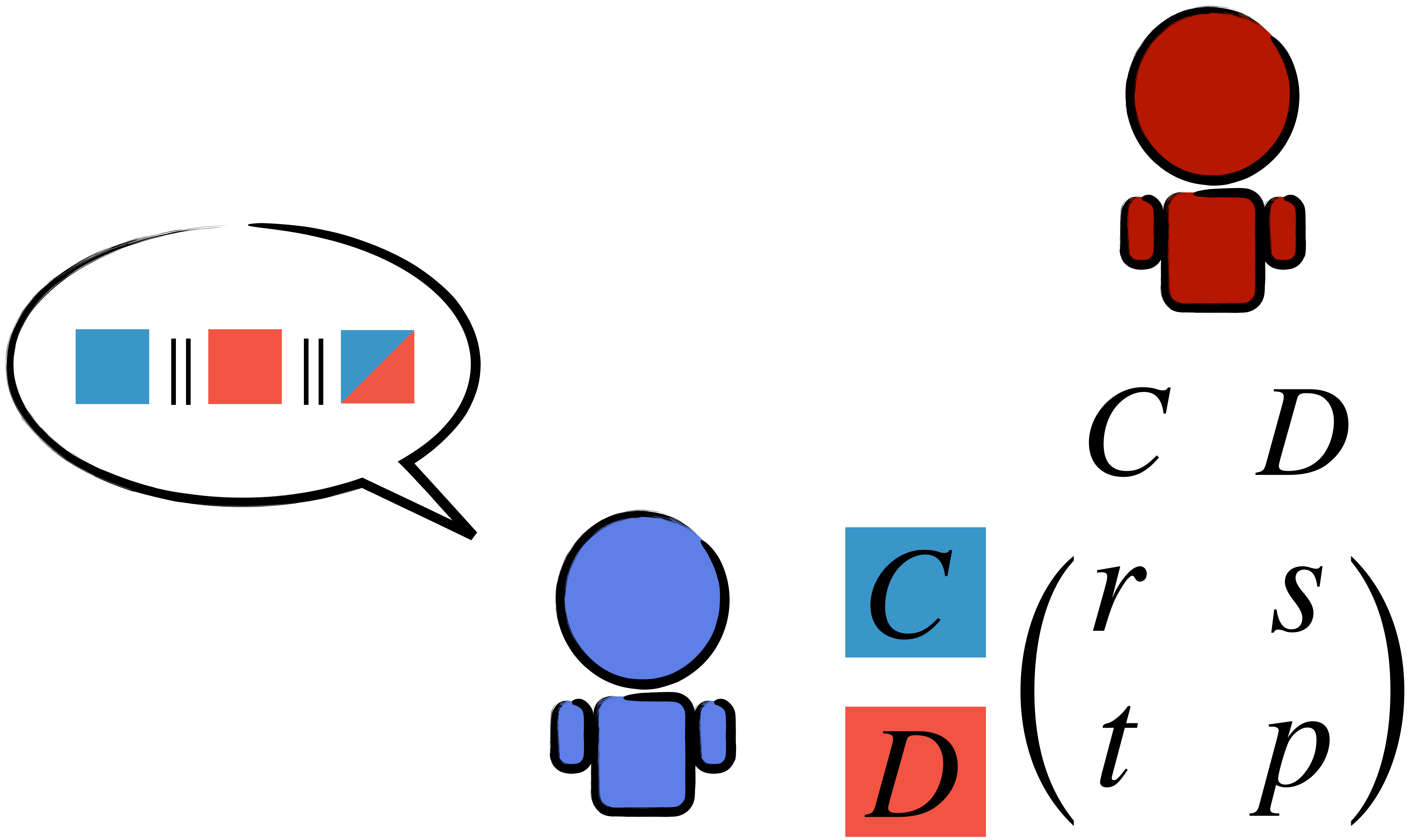
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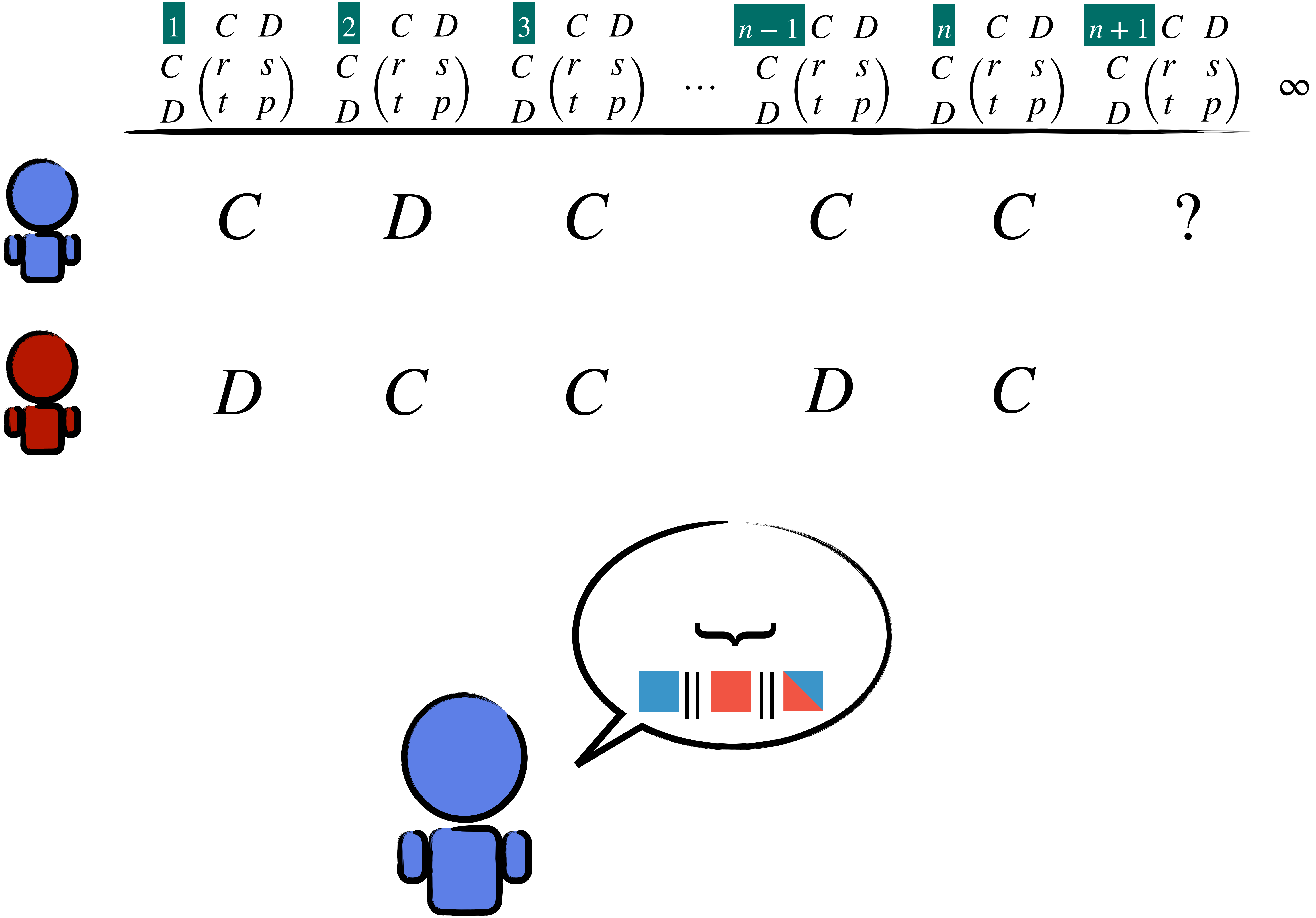
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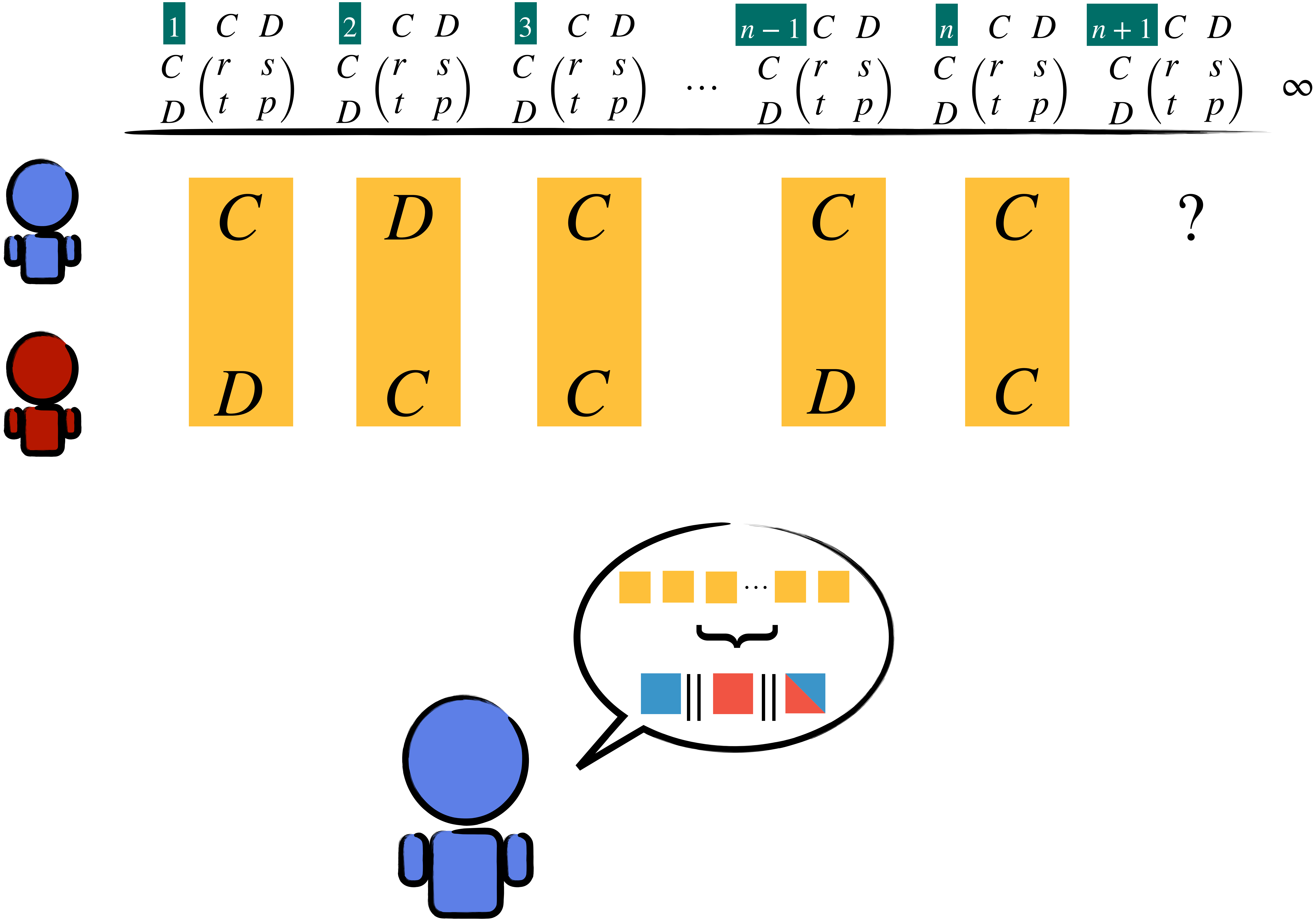

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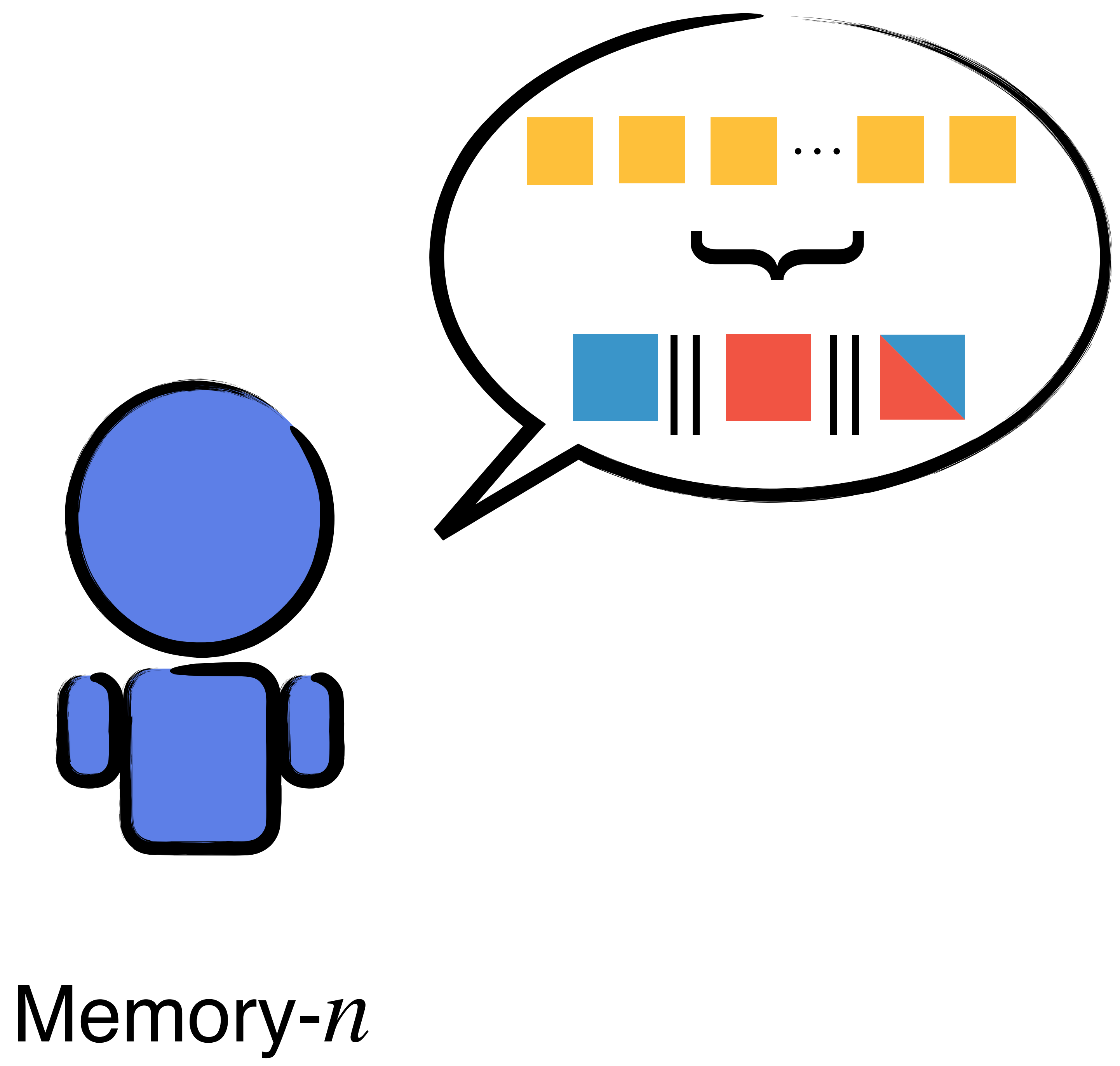
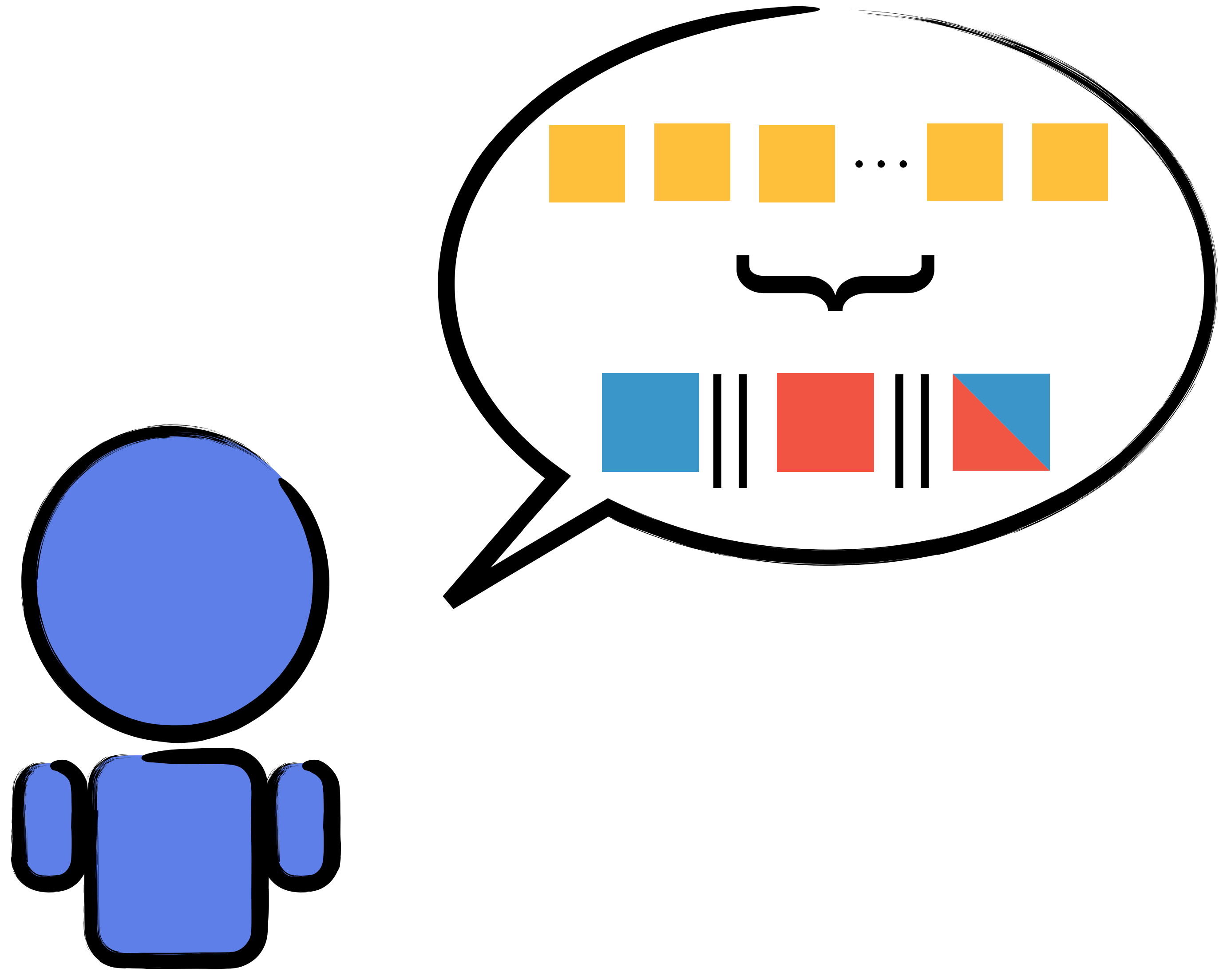
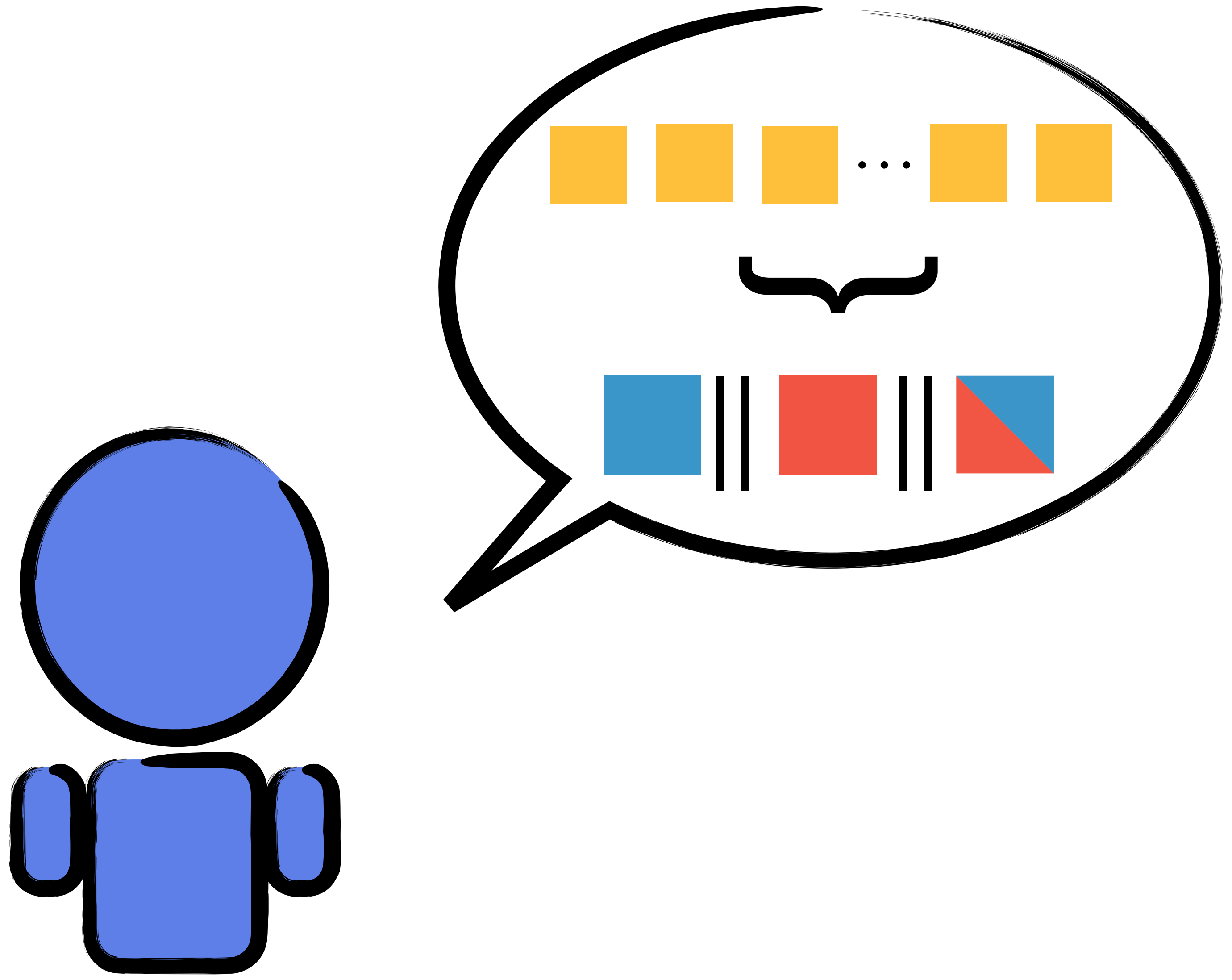

Memory-n



Memory-n

$$\mathbf{m} = (m_{\mathbf{h}})_{\mathbf{h} \in H}$$



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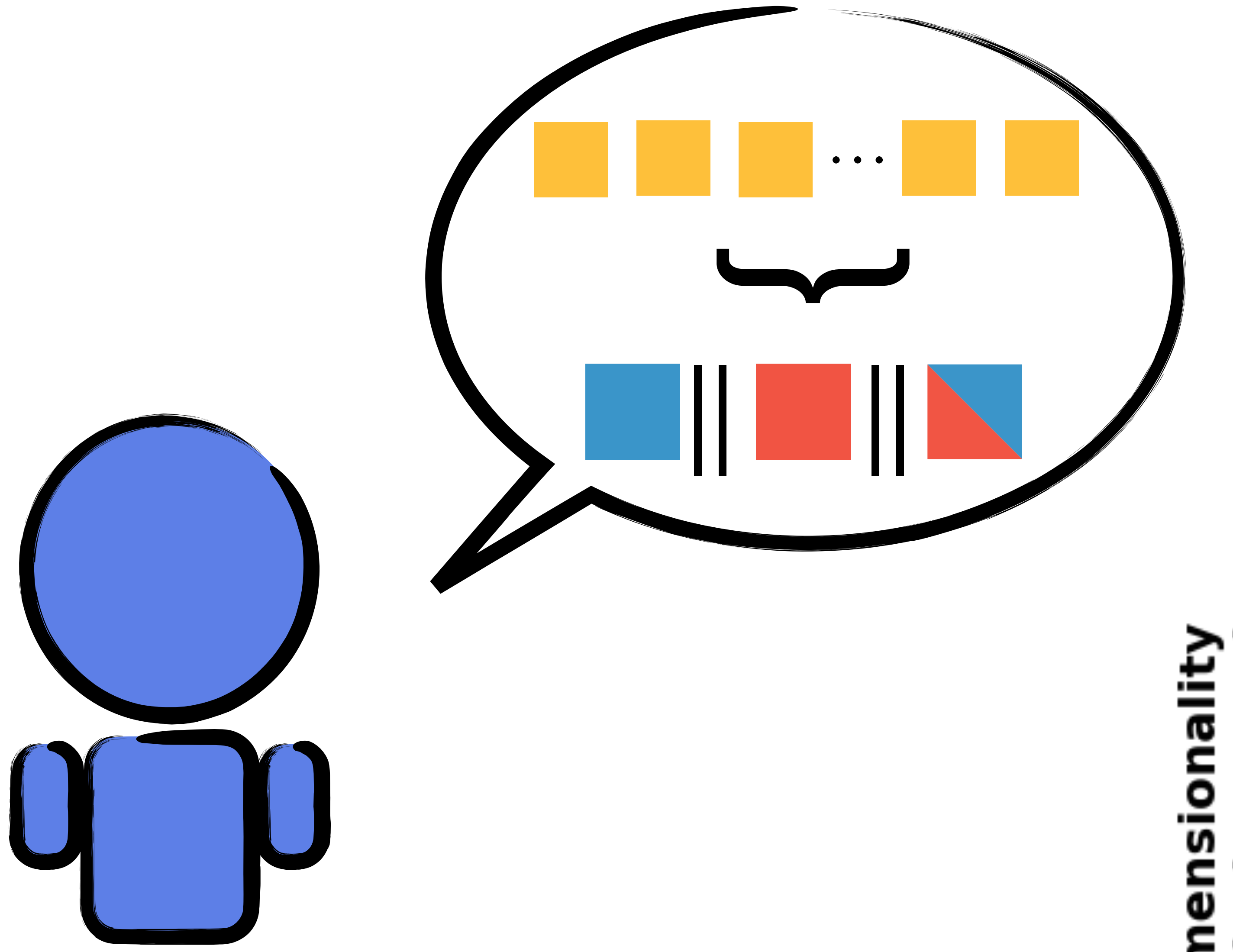
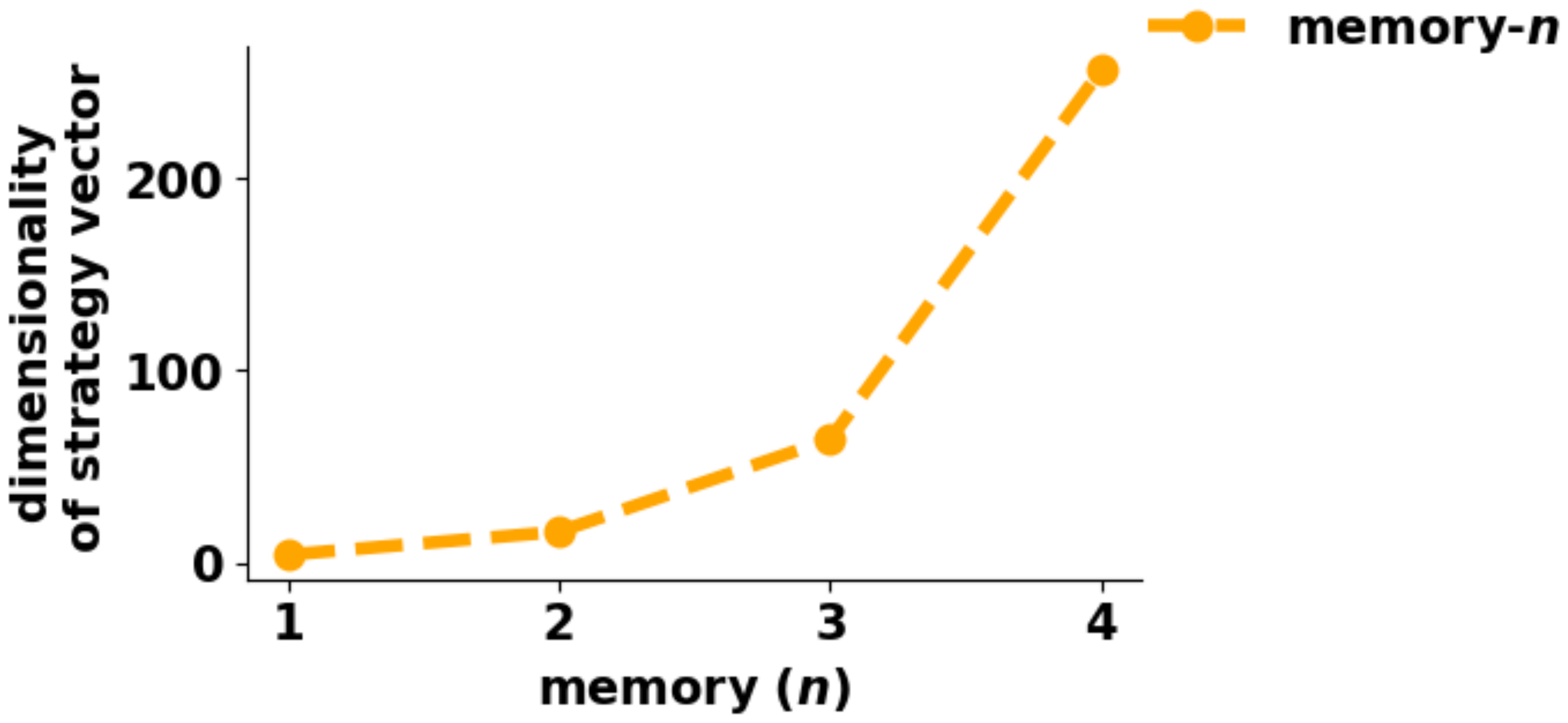
$$[0,1]^{2^{2n}}$$





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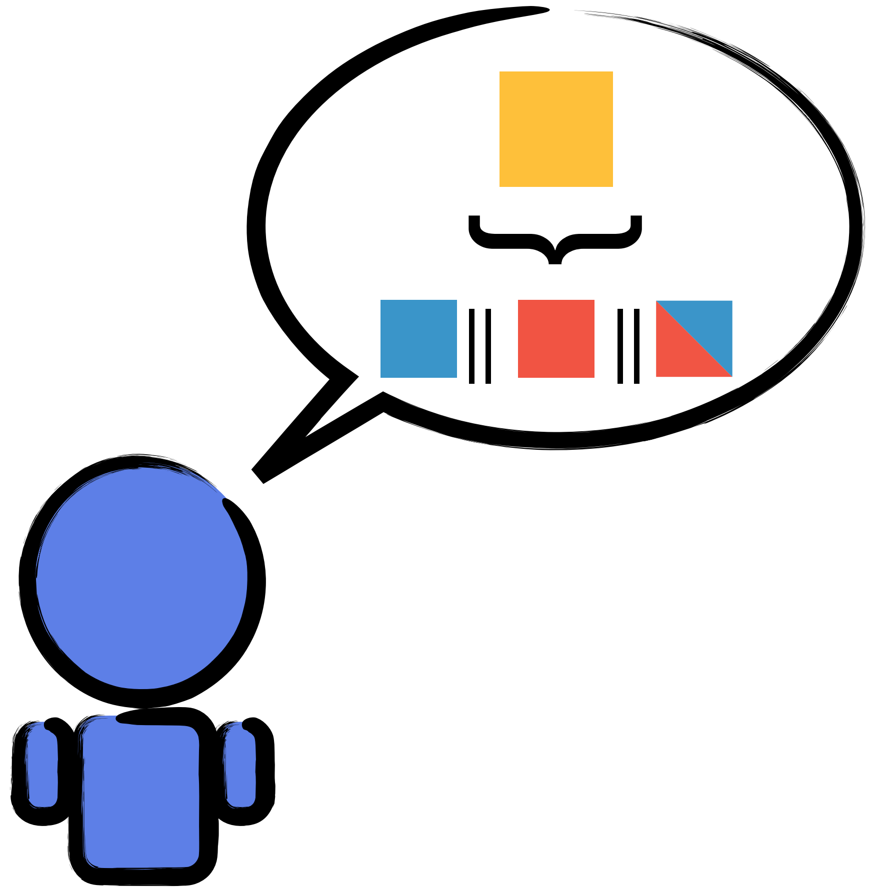

Memory-1

[1] Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent.

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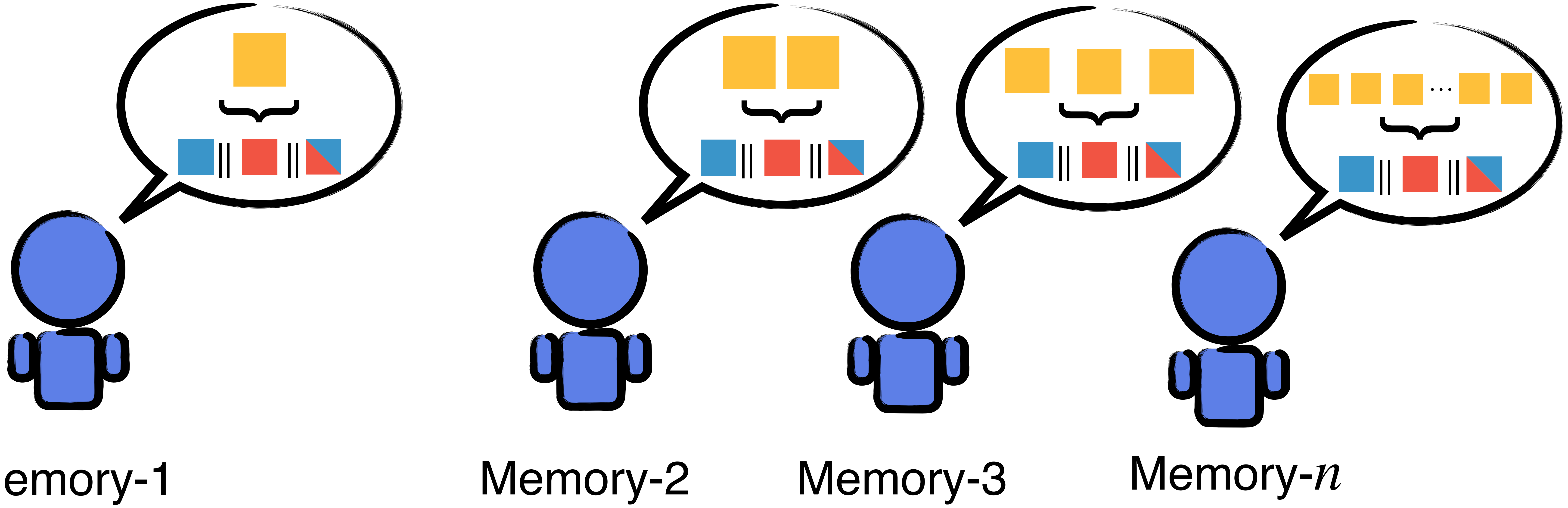

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[3] Hilbe, C., Martinez-Vaquero, L.A., Chatterjee, K. and Nowak, M.A., 2017. Memory$-n$ strategies of direct reciprocity.

[4] Murase, Y. and Baek, S.K., 2023. Grouping promotes both partnership and rivalry with long memory in direct reciprocity.

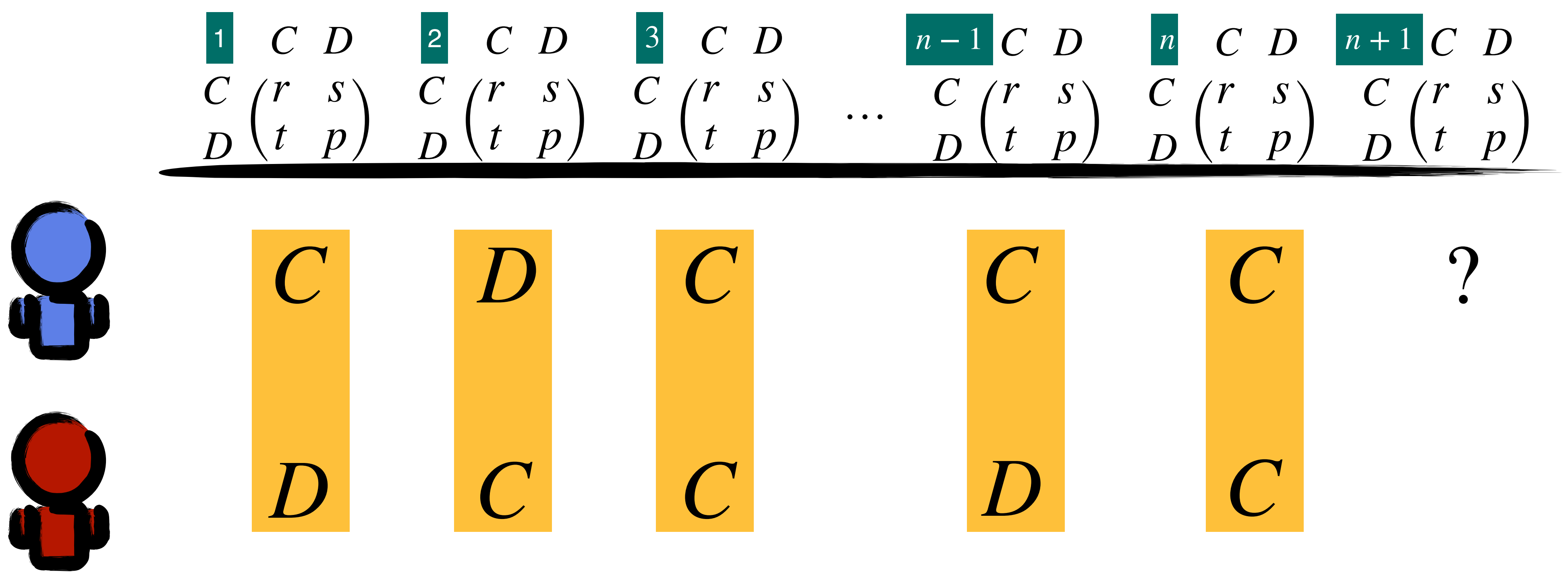
[5] S Do Yi, SK Baek, JK Choi, 2017. Combination with anti-tit-for-tat remedies problems of tit-for-tat.

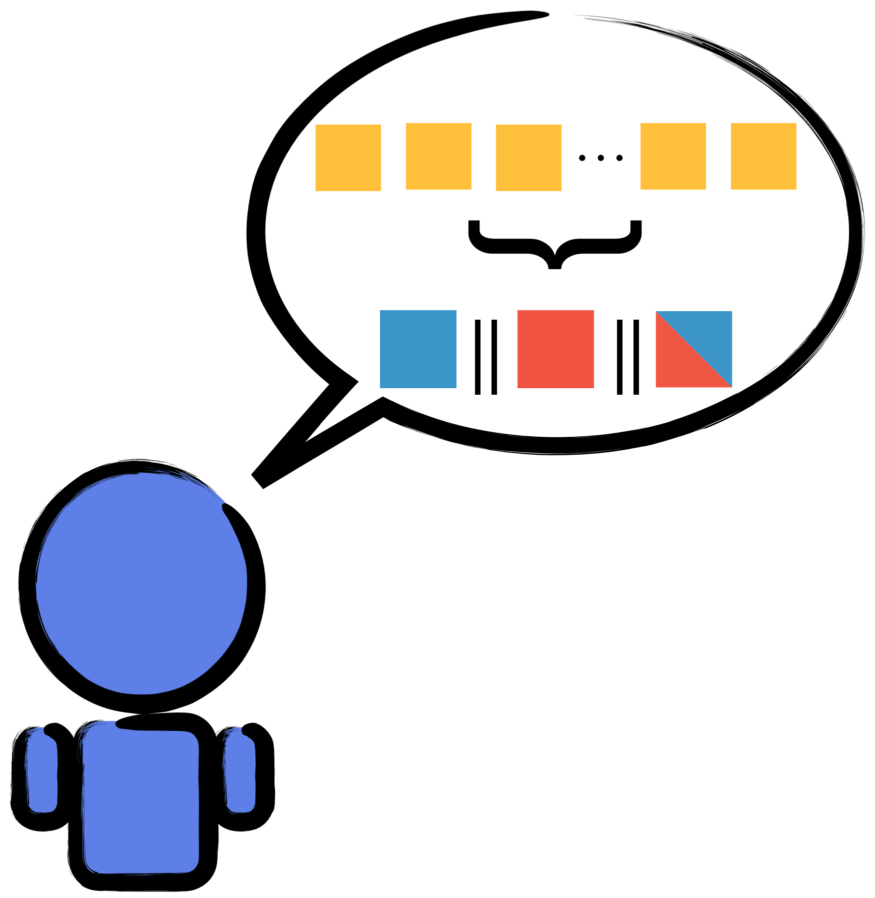
[6] M Ueda, 2021. Memory-two zero-determinant strategies in repeated games.

[7] J Li, et al., 2022. Evolution of cooperation through cumulative reciprocity.

[8] AJ Stewart, JB Plotkin, 2016. Small groups and long memories promote cooperation.

Can we say anything about Nash equilibria in repeated games for any n?





Memory-n

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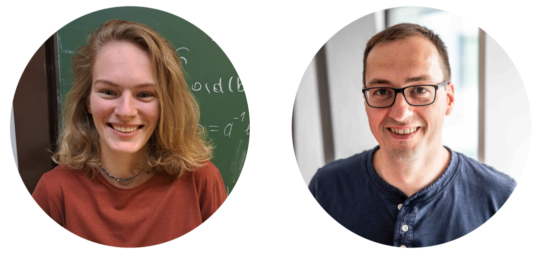

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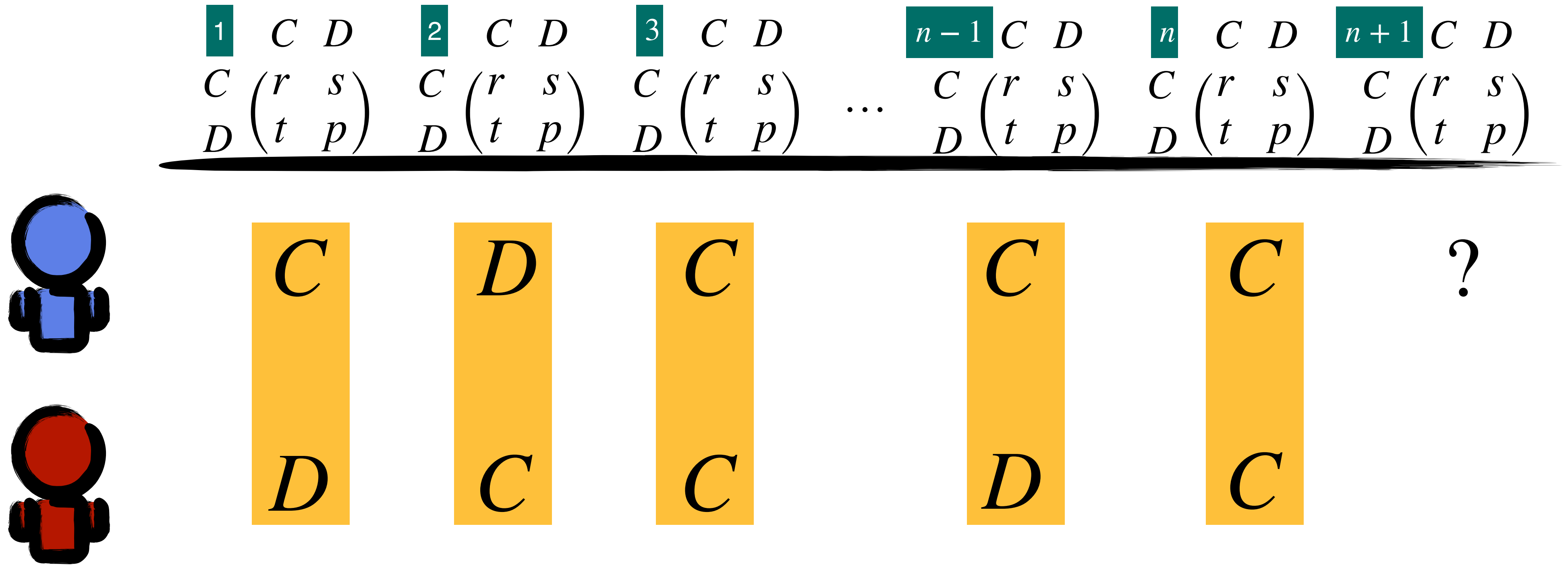

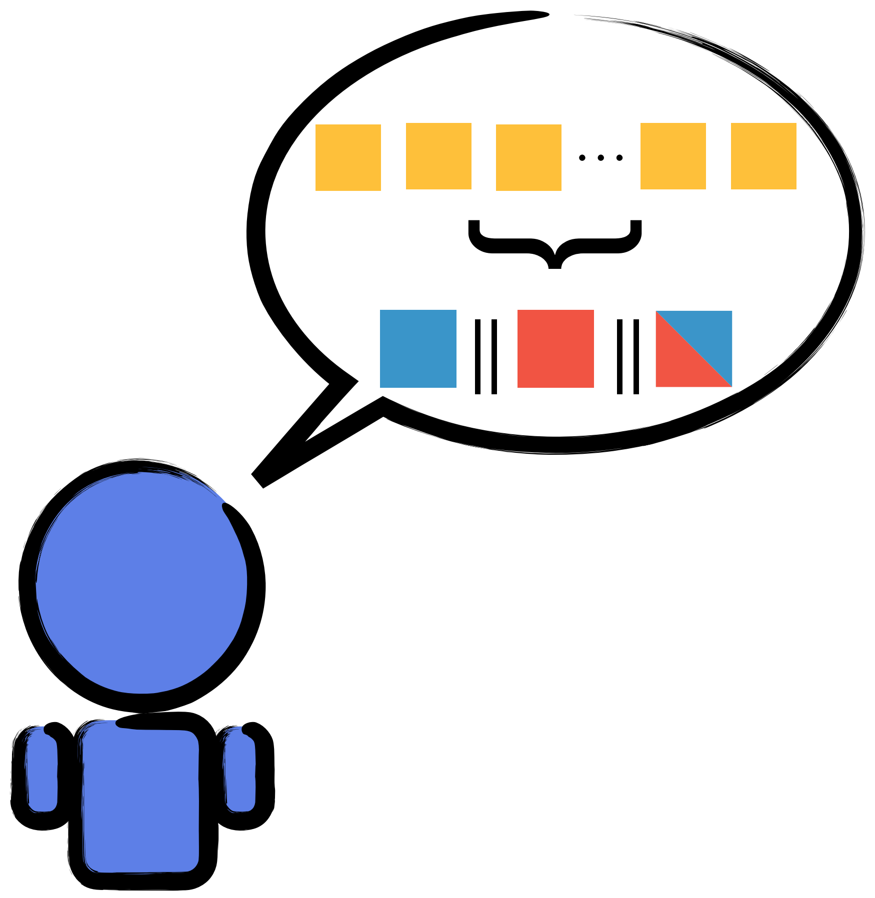

Memory-n

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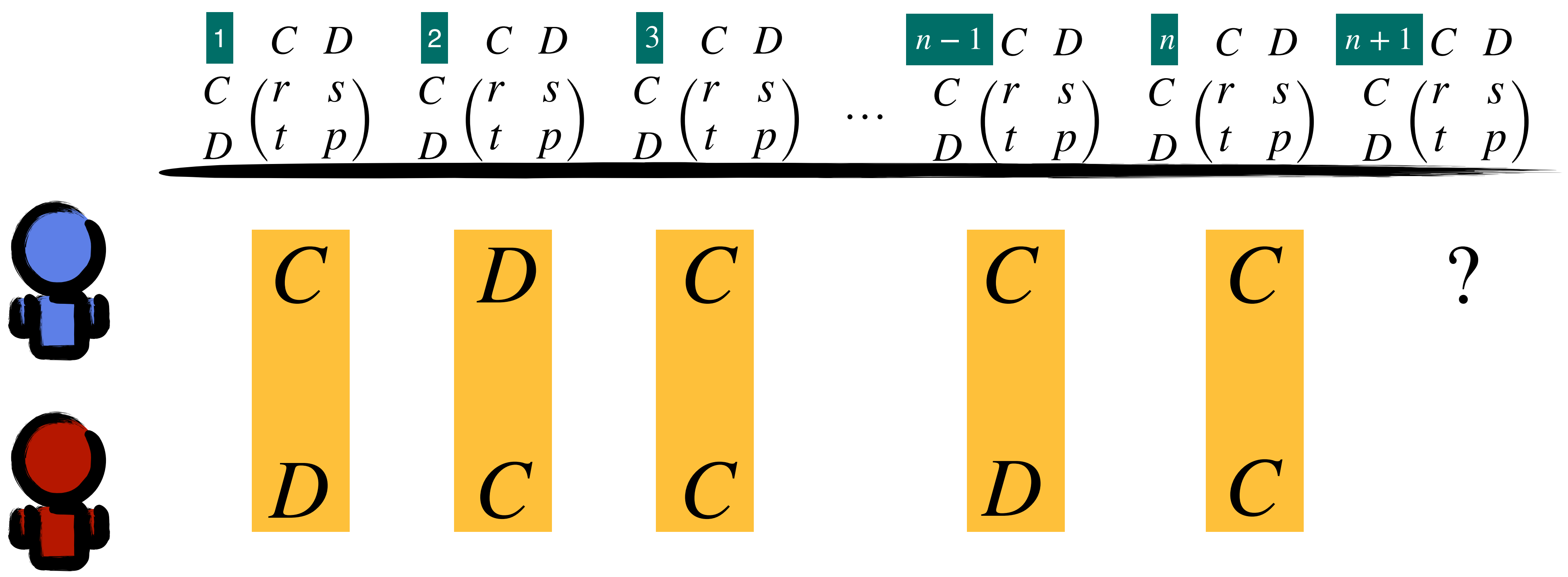

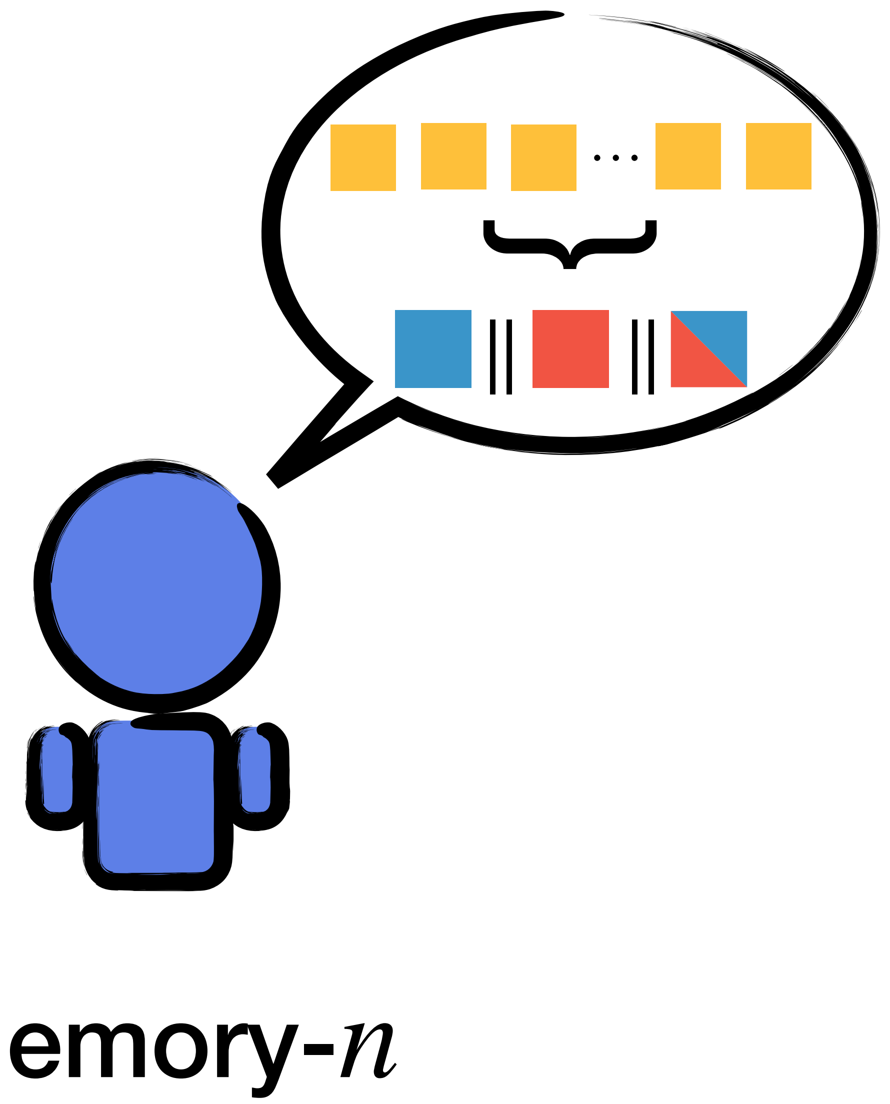

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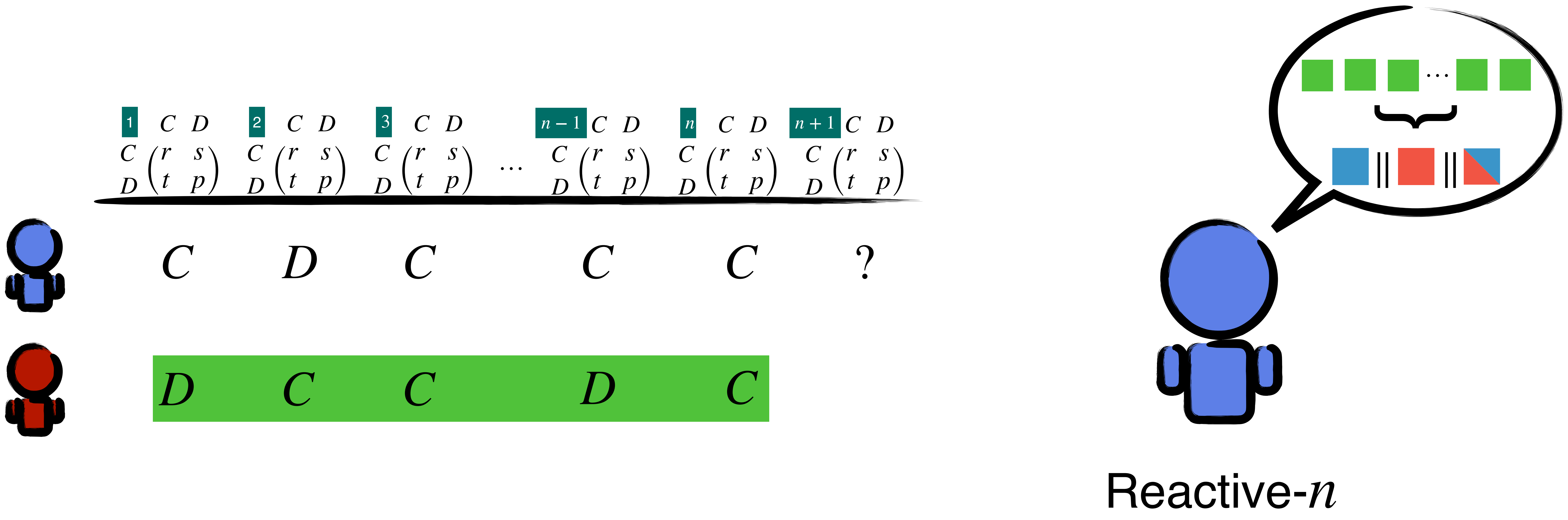

Definitions.

A reactive-n strategy can be defined as 2^n-dimensional vector $\mathbf{p} = (p_{\mathbf{h^{-i}}})_{\mathbf{h^{-i}} \in H^{-i}}$ with $0 \leq p_{\mathbf{h^{-i}}} \leq 1$ where $\mathbf{h^{-i}}$ refers to an n-history of the co-player from the space of all possible co-player histories.

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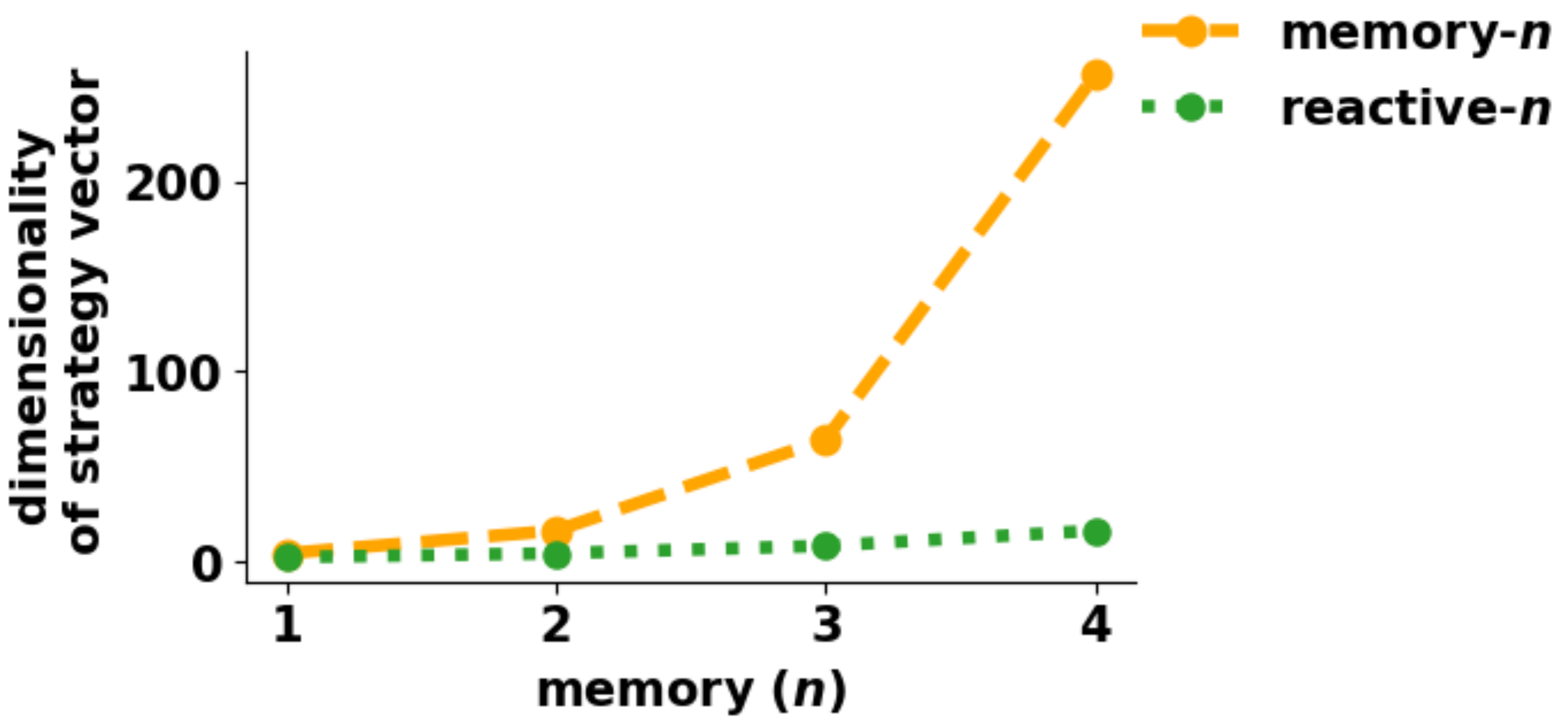

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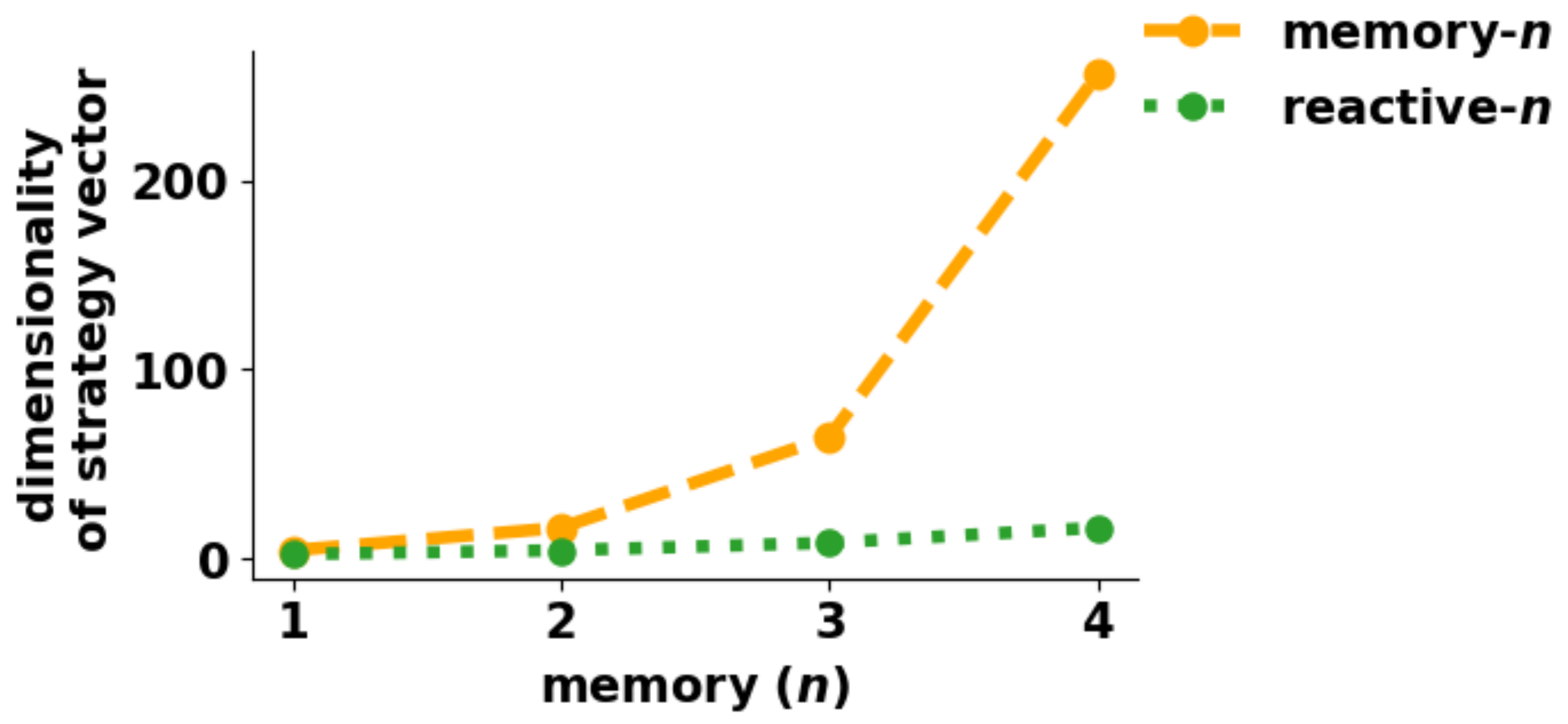
Tit for tat $(1,0)$



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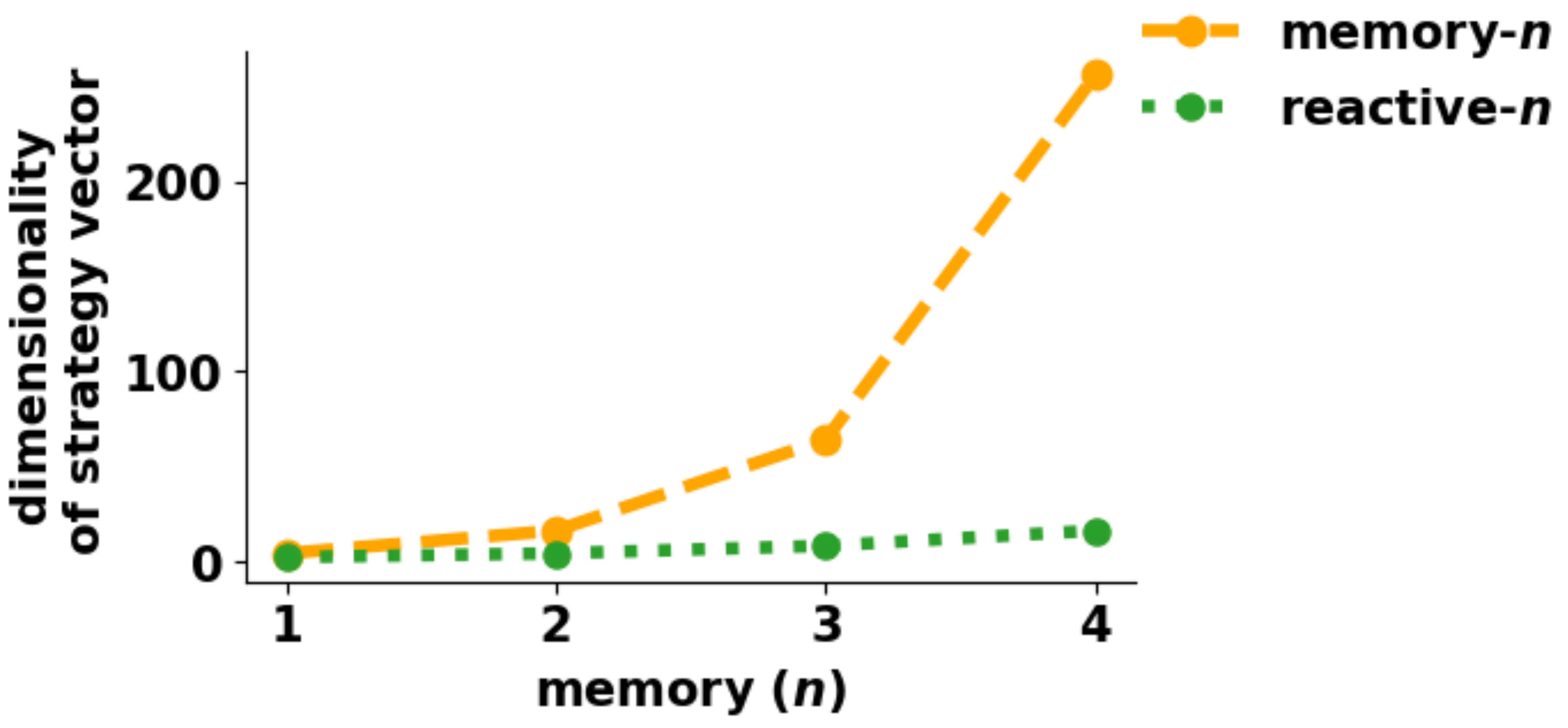
Tit for tat $(1,0)$
Random $(1/2,1/2)$



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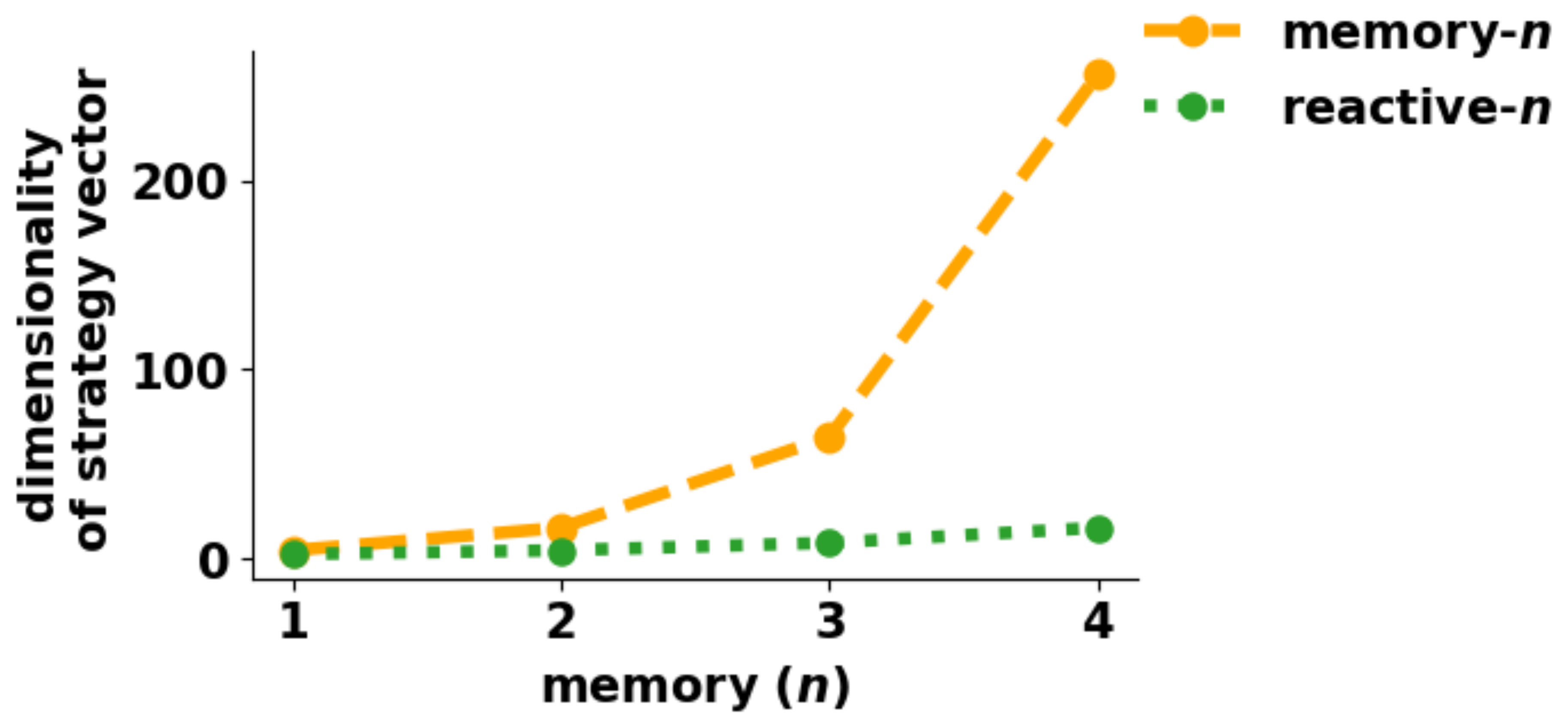
Tit for tat $(1,0)$
Random $(1/2,1/2)$
Two for Two Tats $(1,1,1,0)$



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Examples.

Tit for tat $(1,0) \leftarrow$ pure
Random $(1/2,1/2) \leftarrow$ stochastic
Two for Two Tats $(1,1,1,0) \leftarrow$ pure

Definitions.

A strategy is considered pure if all conditional cooperation probabilities are either zero or one. If all cooperation probabilities are strictly between zero and one, the strategy is described as stochastic.

Examples.

Tit for tat $(1,0) \leftarrow$ pure
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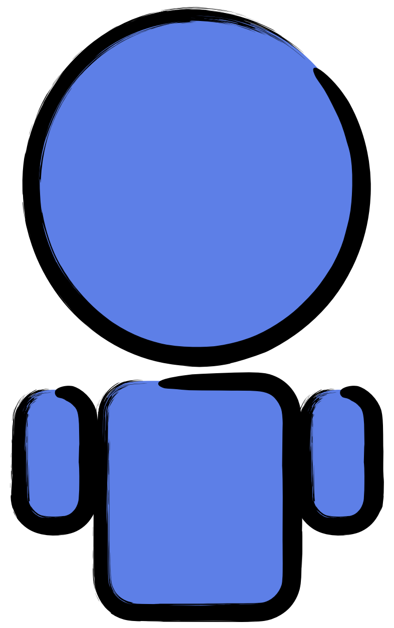
Two for Two Tats $(1,1,1,0) \leftarrow$ pure

Nash Definition.

A strategy $\mathbf{p}$ for a repeated game is a Nash equilibrium if it is a best response to itself.
That is $\pi(\mathbf{p},\mathbf{p}) \geq \pi(\sigma,\mathbf{p})$ for all other strategies σ.



p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memory-n strategies σ.

[1] Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent.


p

Nash equilibria in repeated games for any n when blue is playing as reactive-n?

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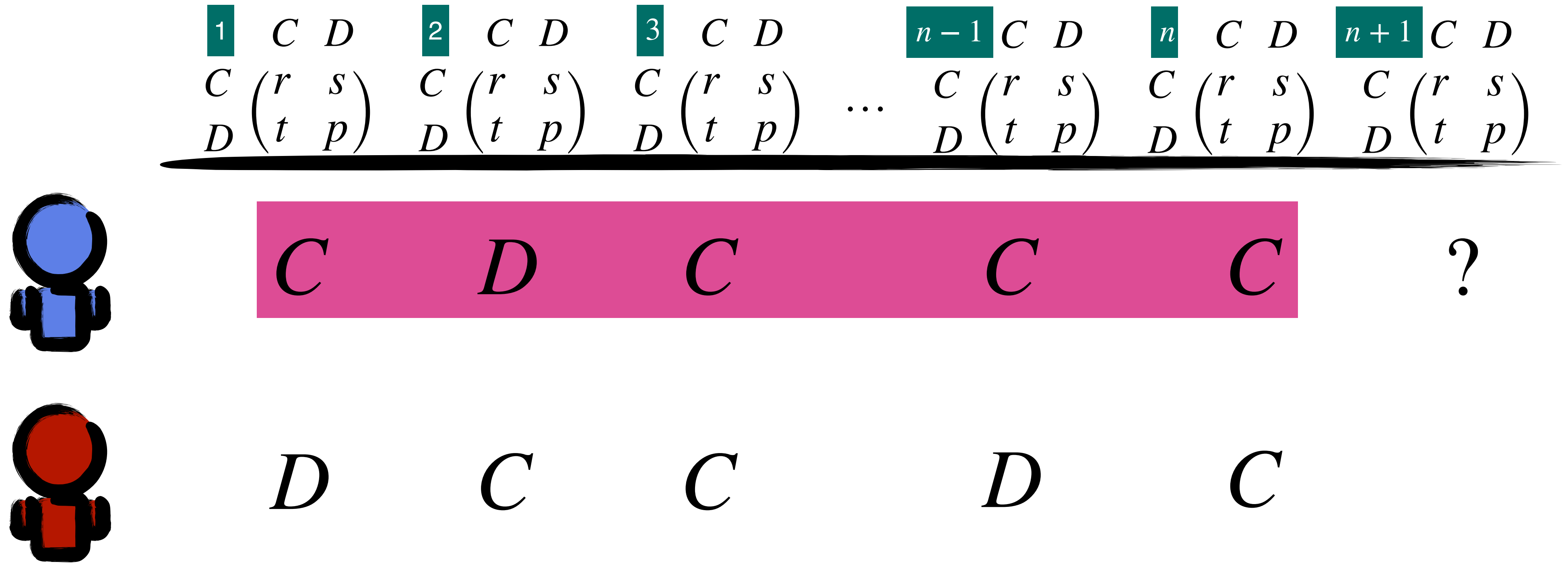
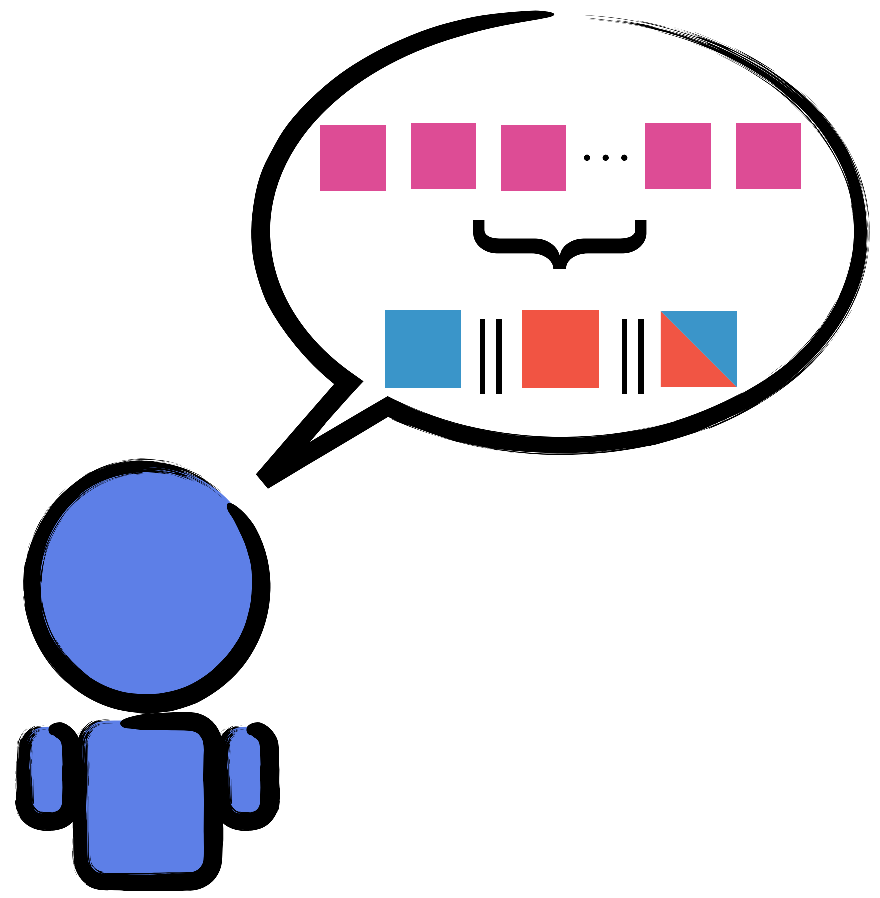
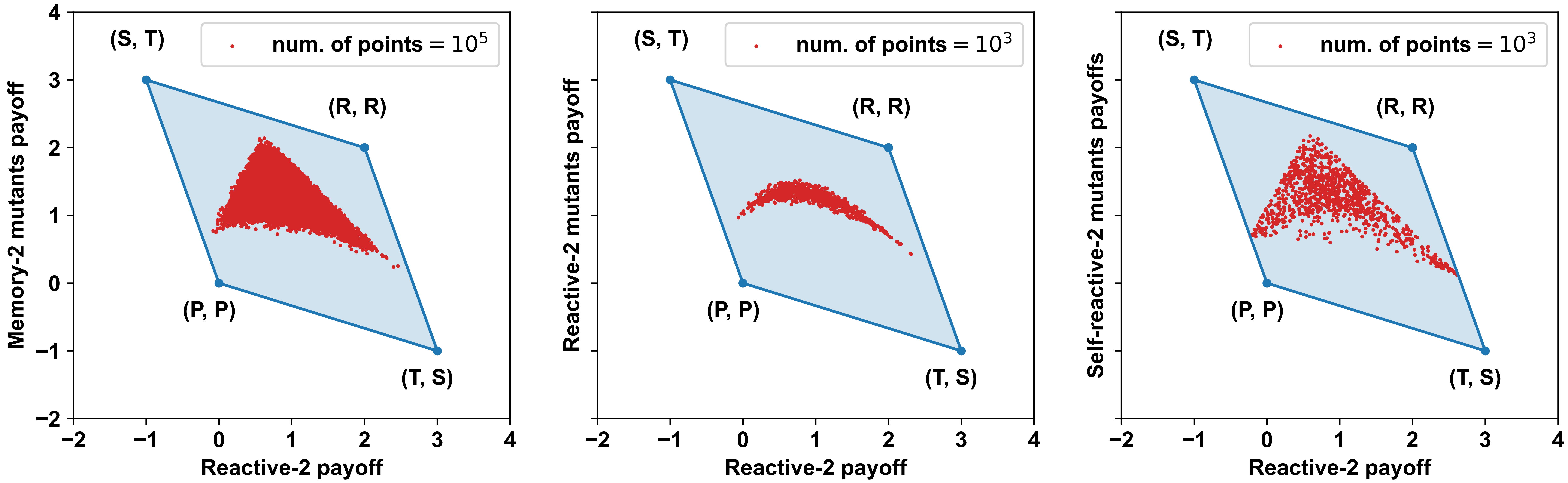




Self-reactive-n



num. of points = 10^5
(S, T)
(R, R)
(P, P)
(T, S)
Memory-2 mutants payoff
Reactive-2 payoff
num. of points = 10^3
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1. Against reactive strategies, any feasible payoff can be generated with self-reactive strategies.

2. To any reactive strategy, there is a best response among the pure self-reactive strategies.

Theorem. A reactive strategy $\mathbf{p} \in \mathscr{R}_n$ is a Nash equilibrium if and only if $\pi(\mathbf{p}, \mathbf{p}) \geq \pi(\tilde{\mathbf{p}}, \mathbf{p})$ for all pure self-reactive strategies $\tilde{\mathbf{p}}$.

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$\mathbf{p} = (p_C, p_D)$ 4

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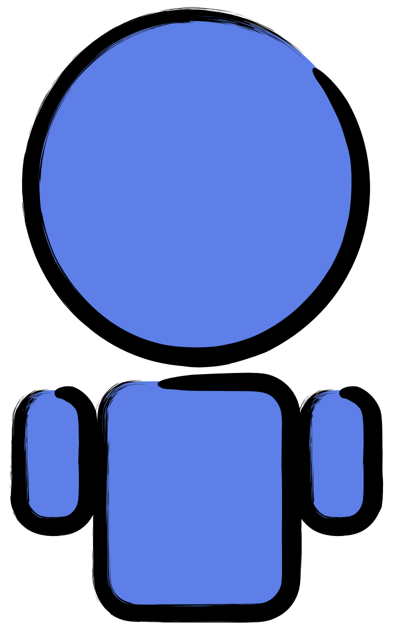
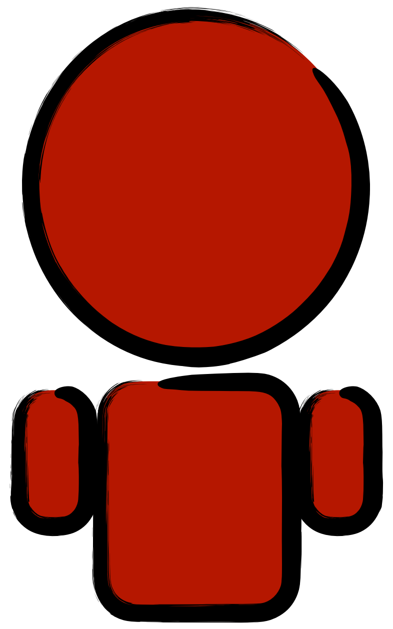
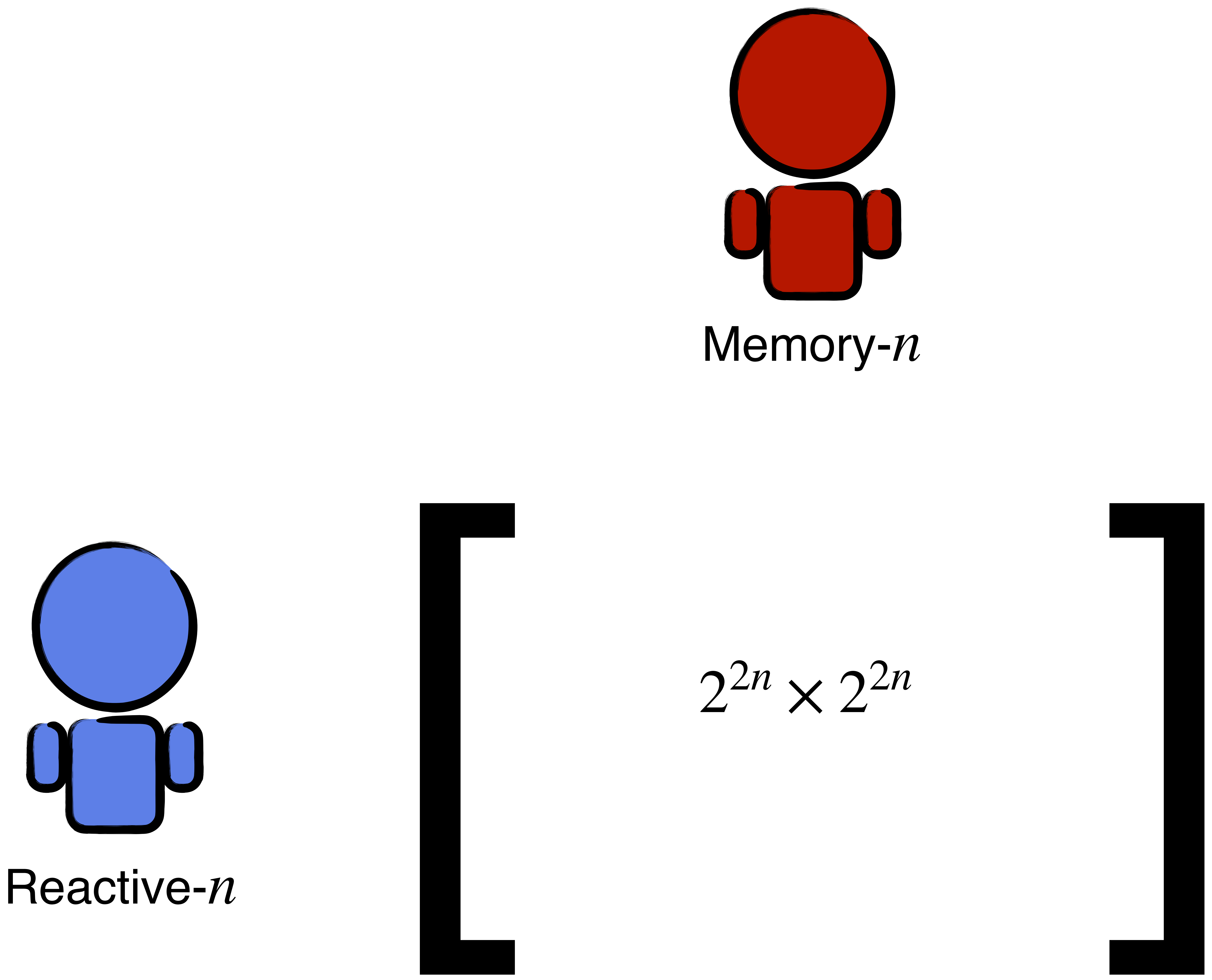
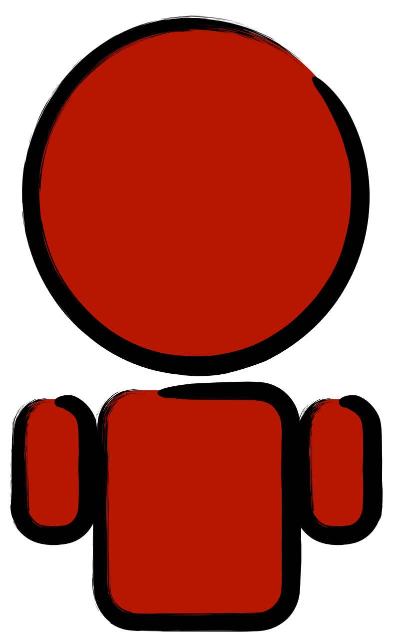




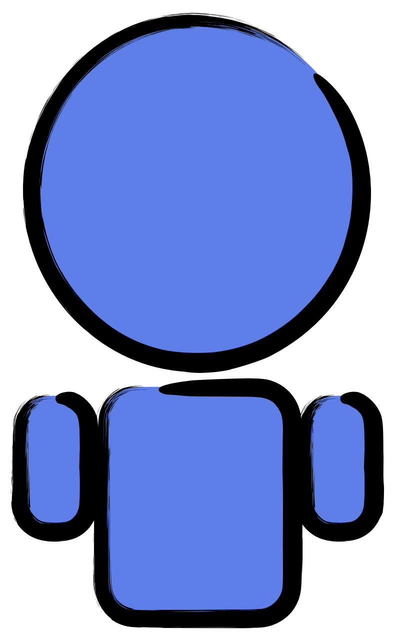
$\mathbf{p} = (p_{CCC}, p_{CCD}, p_{CDC}, p_{CDD}, p_{DCC}, p_{DCD}, p_{DDC}, p_{DDD})$ 256


Memory-n
Reactive-n
$2^{2n} \times 2^{2n}$



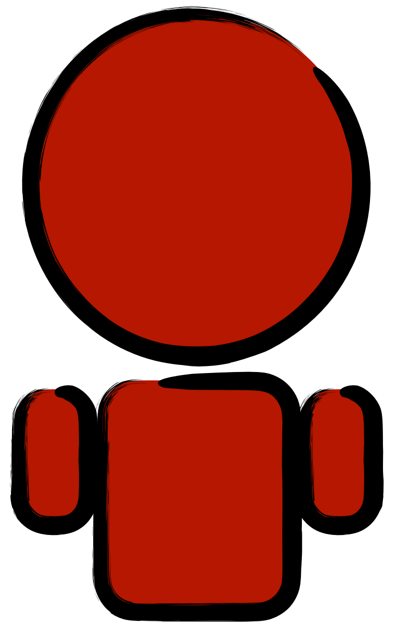
~~Memory-n~~

Self-reactive—n



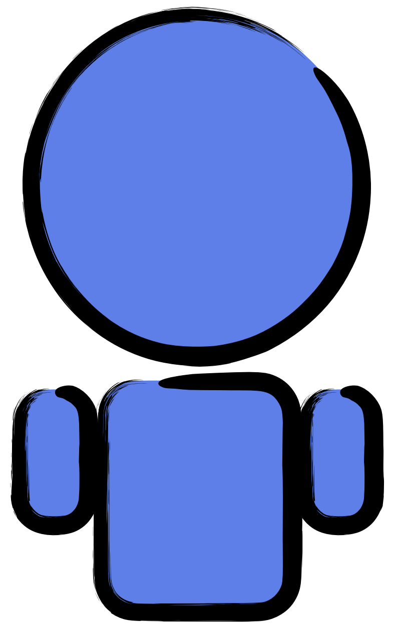
Reactive-n

$$\begin{bmatrix} 2^{2n} \times 2^{2n} \end{bmatrix}$$



~~Memory n~~

Self-reactive—n



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input: p, n

pure_self_reactive_strategies $\leftarrow \left\{ \tilde{\mathbf{p}} \;\middle|\; \tilde{\mathbf{p}} \in \{0,1\}^{2^n} \right\}$;

isNash ←True;

for $\tilde{\mathbf{p}} \in$ *pure_self_reactive_strategies* **do**

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return (**p**, isNash);

Donation game

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array}$$

$$b > c > 0$$

Cooperative Nash

Cooperative Nash

Theorem. A reactive-2 strategy $\mathbf{p}$ is a cooperative Nash equilibrium if and only if its entries satisfy the conditions,

$$p_{CC} = 1, \quad \frac{p_{CD} + p_{DC}}{2} \leq 1 - \frac{1}{2} \cdot \frac{c}{b}, \quad p_{DD} \leq 1 - \frac{c}{b}.$$

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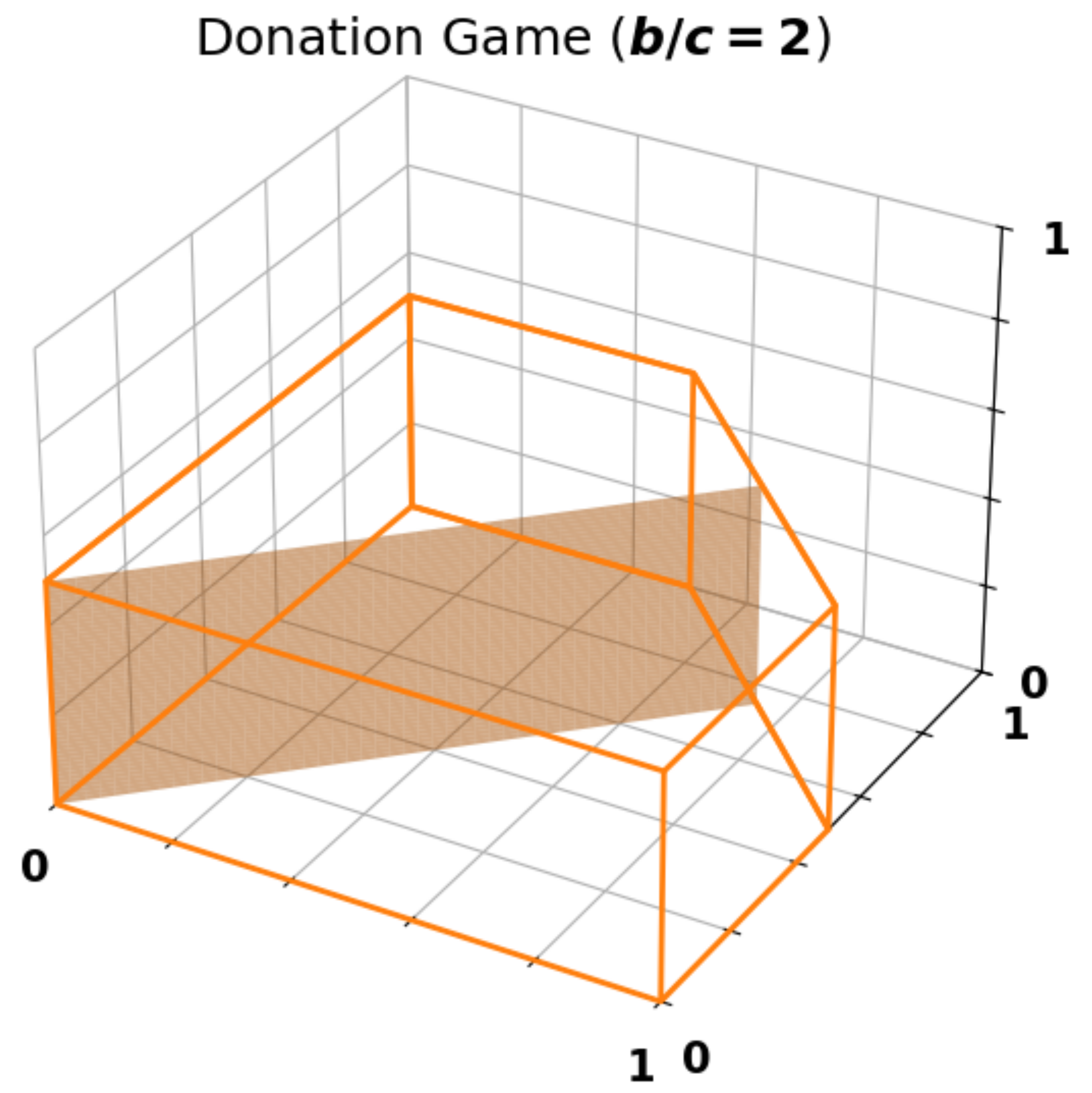

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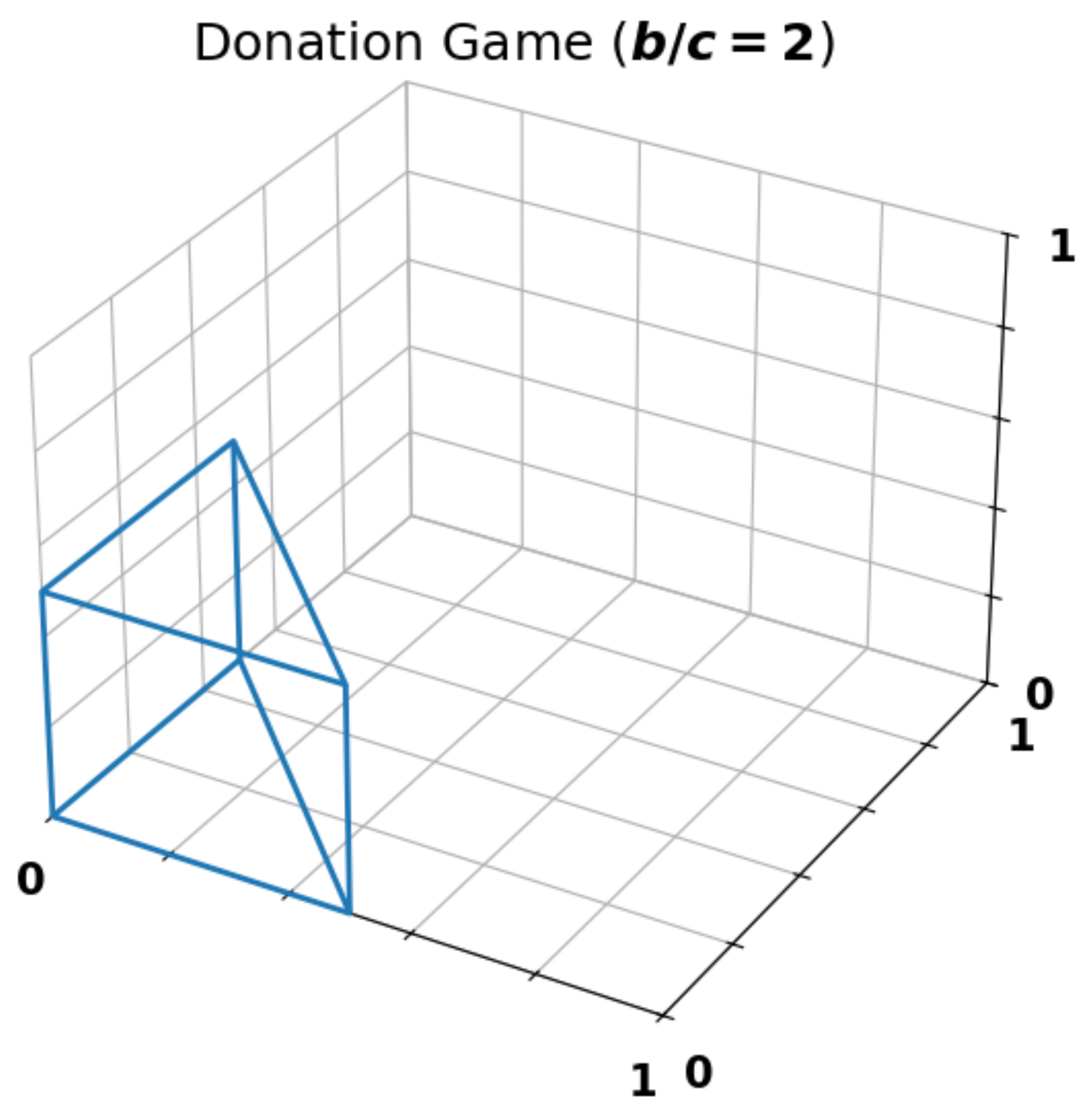

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$$\frac{p_{CCD} + p_{CDD} + p_{DCC} + p_{DDC}}{4} \leq 1 - \frac{1}{2} \cdot \frac{c}{b} \qquad p_{DDD} \leq 1 - \frac{c}{b}$$

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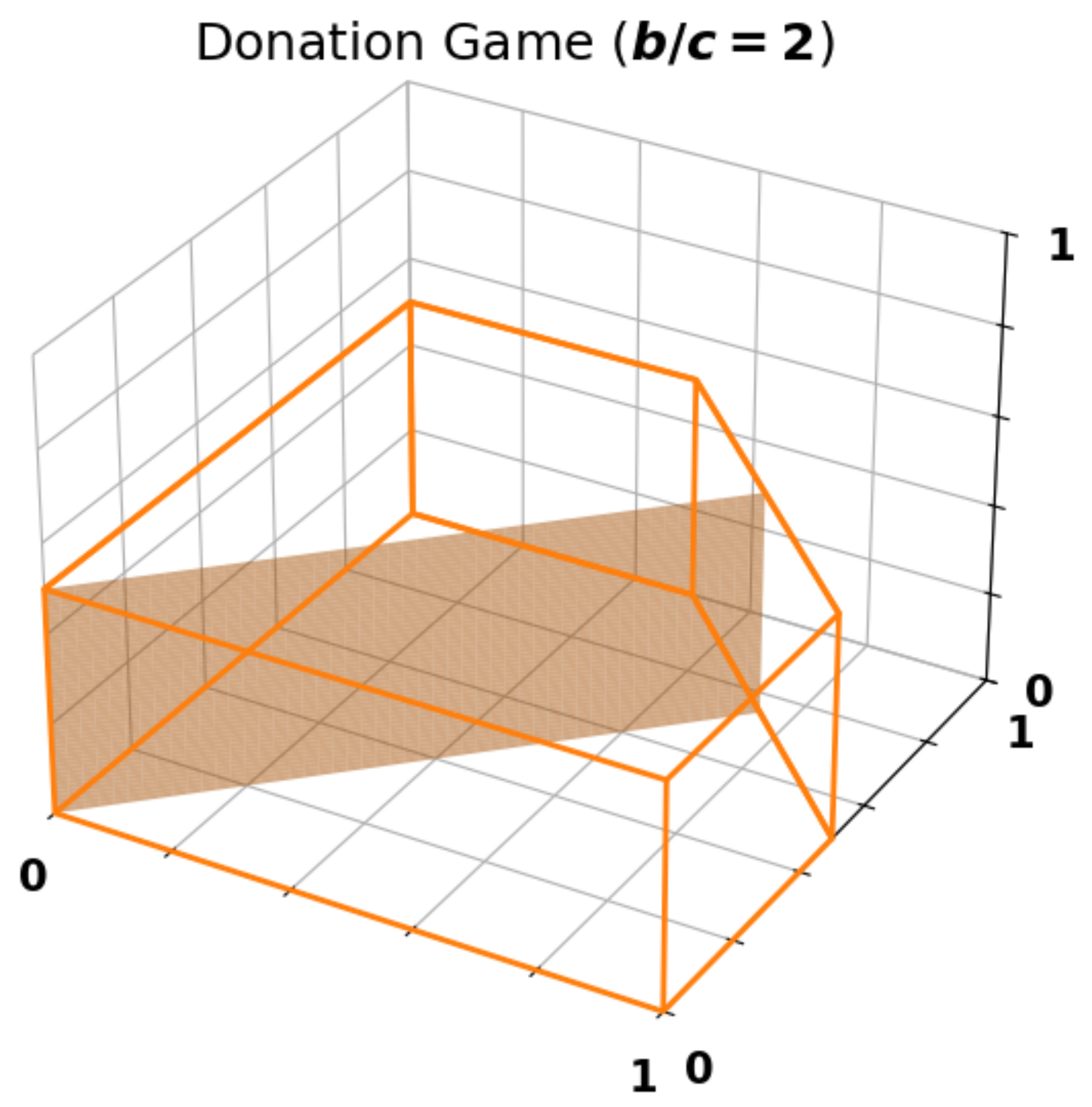
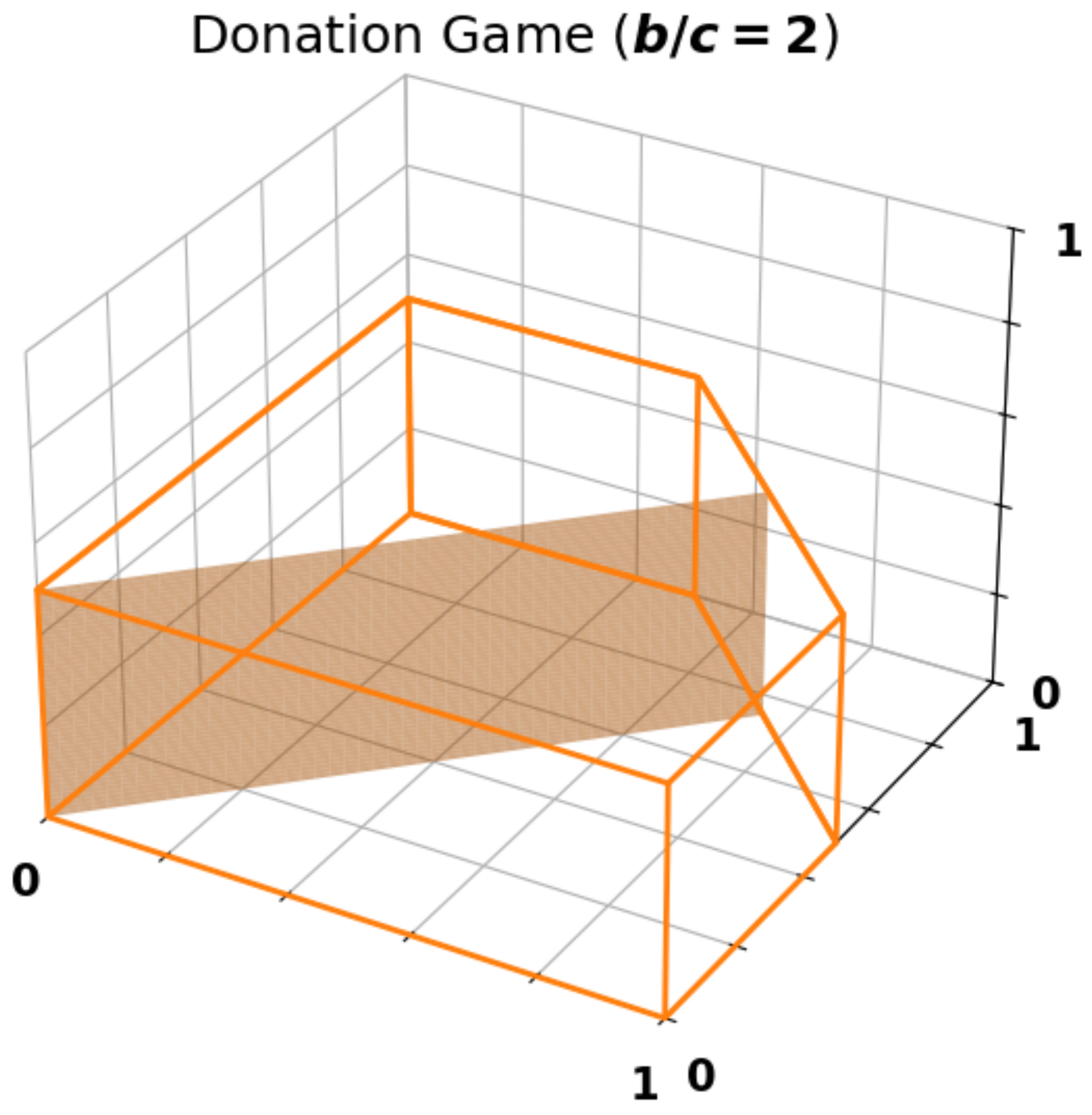
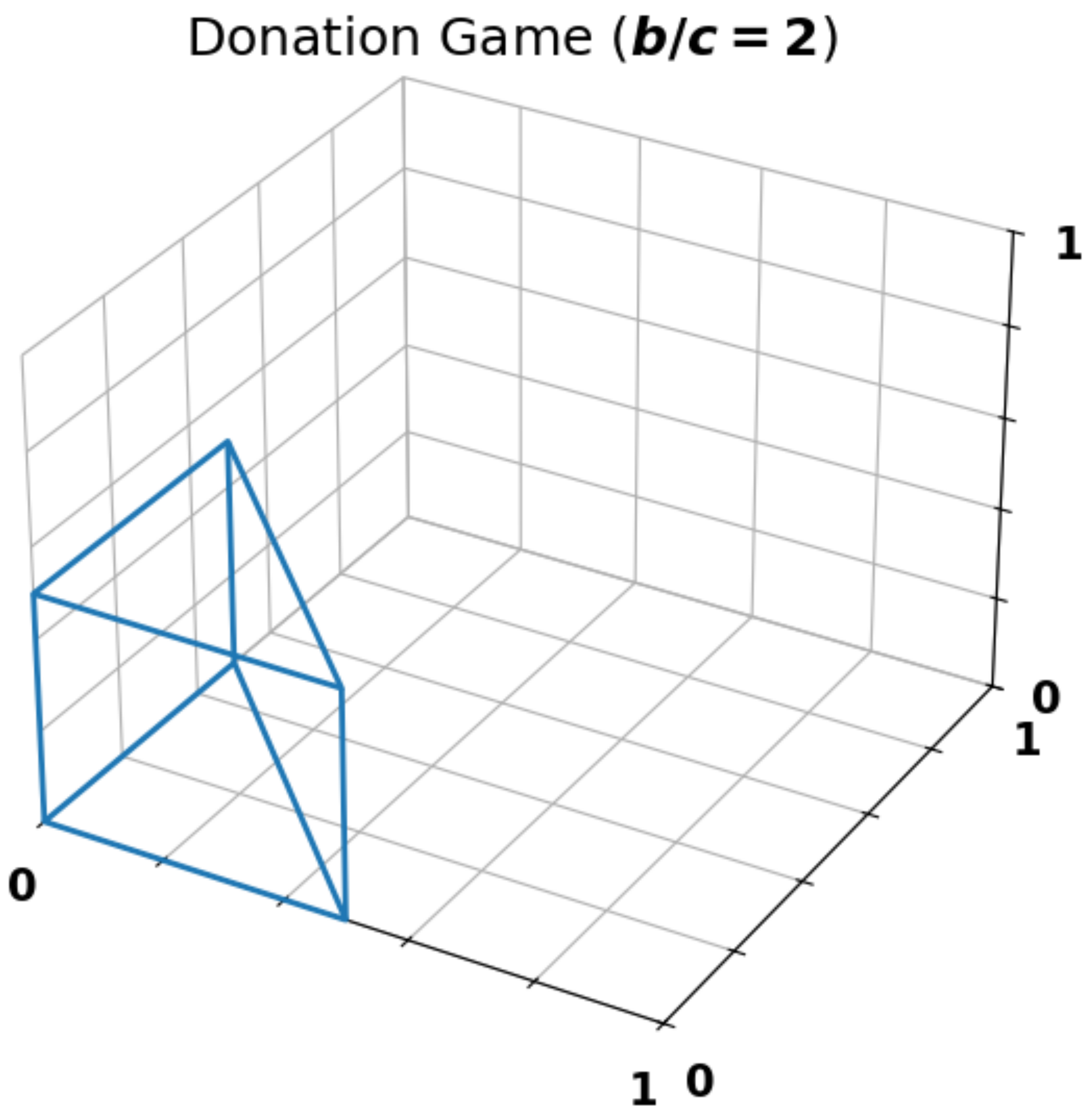
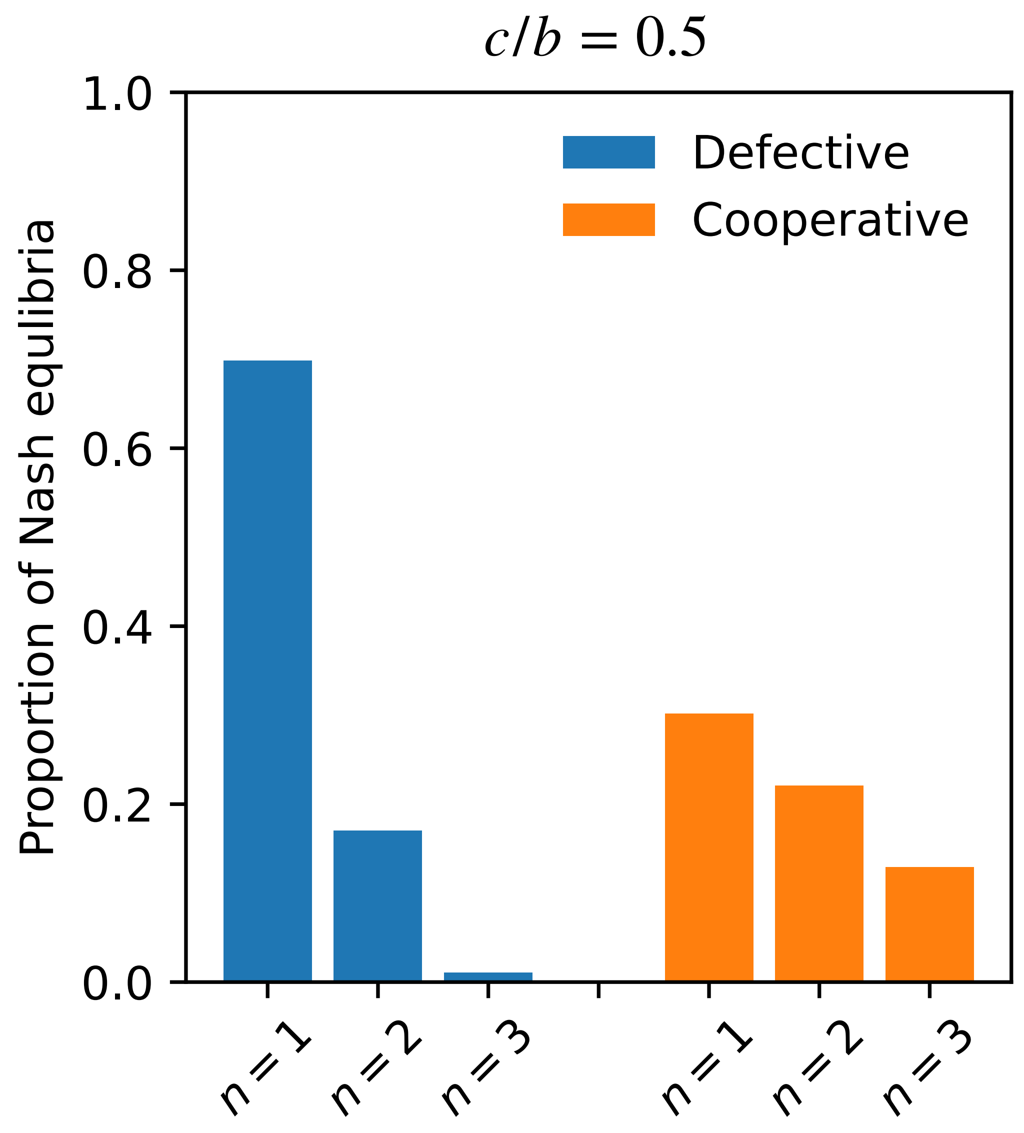

Donation Game (b/c = 2)
0
1
0
1
1
0


Donation Game ($b/c = 2$)
0
1
0
1
0
1


Donation Game ($b/c = 2$)
0
1
0
1
0
1

Cooperative & Defective Nash



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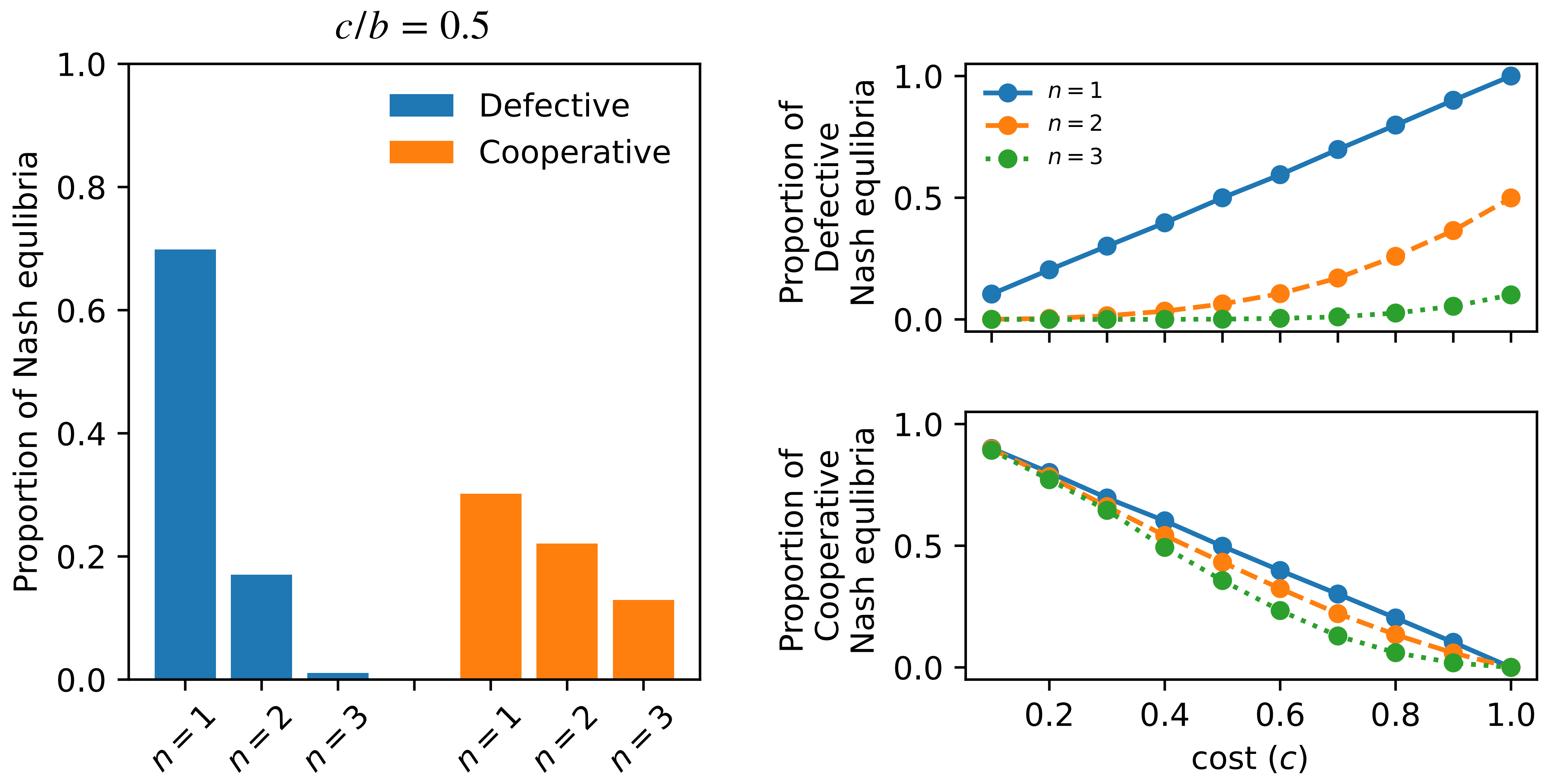

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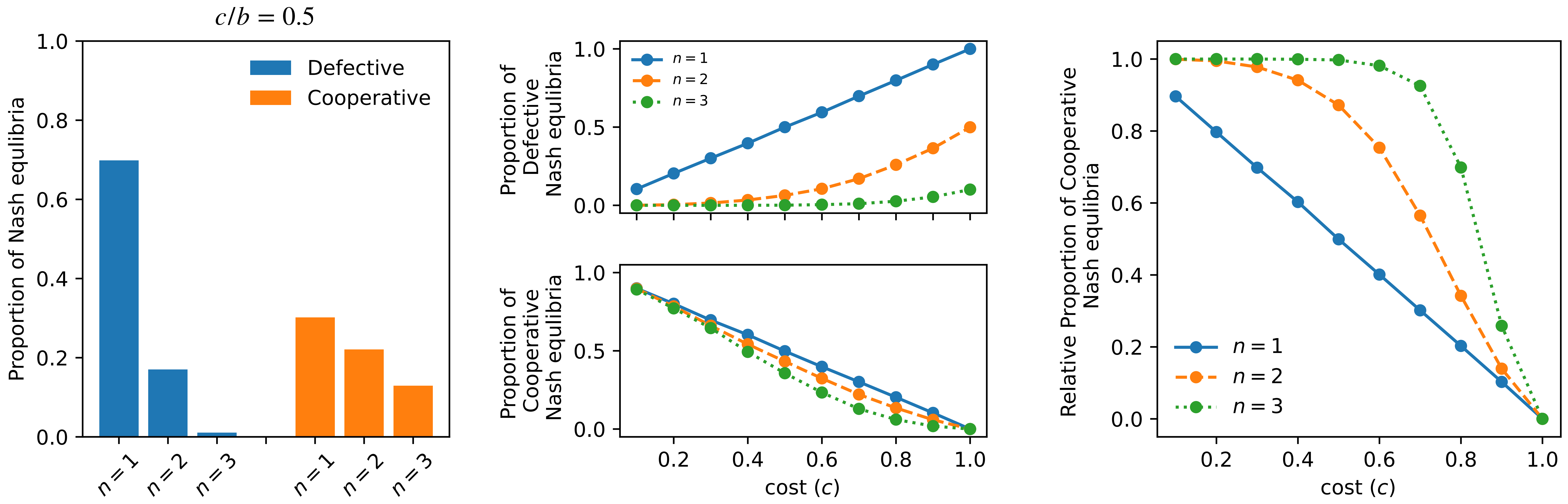

Errors

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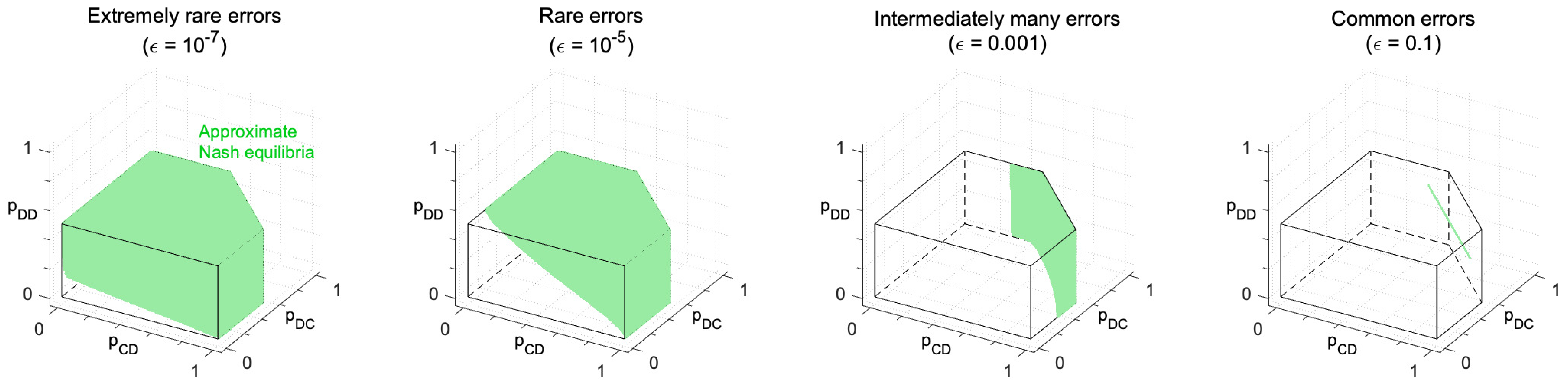

Reactive counting strategies

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A reactive-n counting strategy records how often the co-player has cooperated during the last n rounds.

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$$r_n=1 \qquad \textit{and} \qquad r_{n-k} \leq 1-\frac{k}{n}\cdot\frac{c}{b} \quad \text{for } k \in \{1,2,\ldots,n\}\,.$$

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Theorem. A reactive-n counting strategy $\mathbf{r}=(r_k)_{k\in\{n,n-1,\ldots,0\}}$, is a defective Nash equilibrium if and only if

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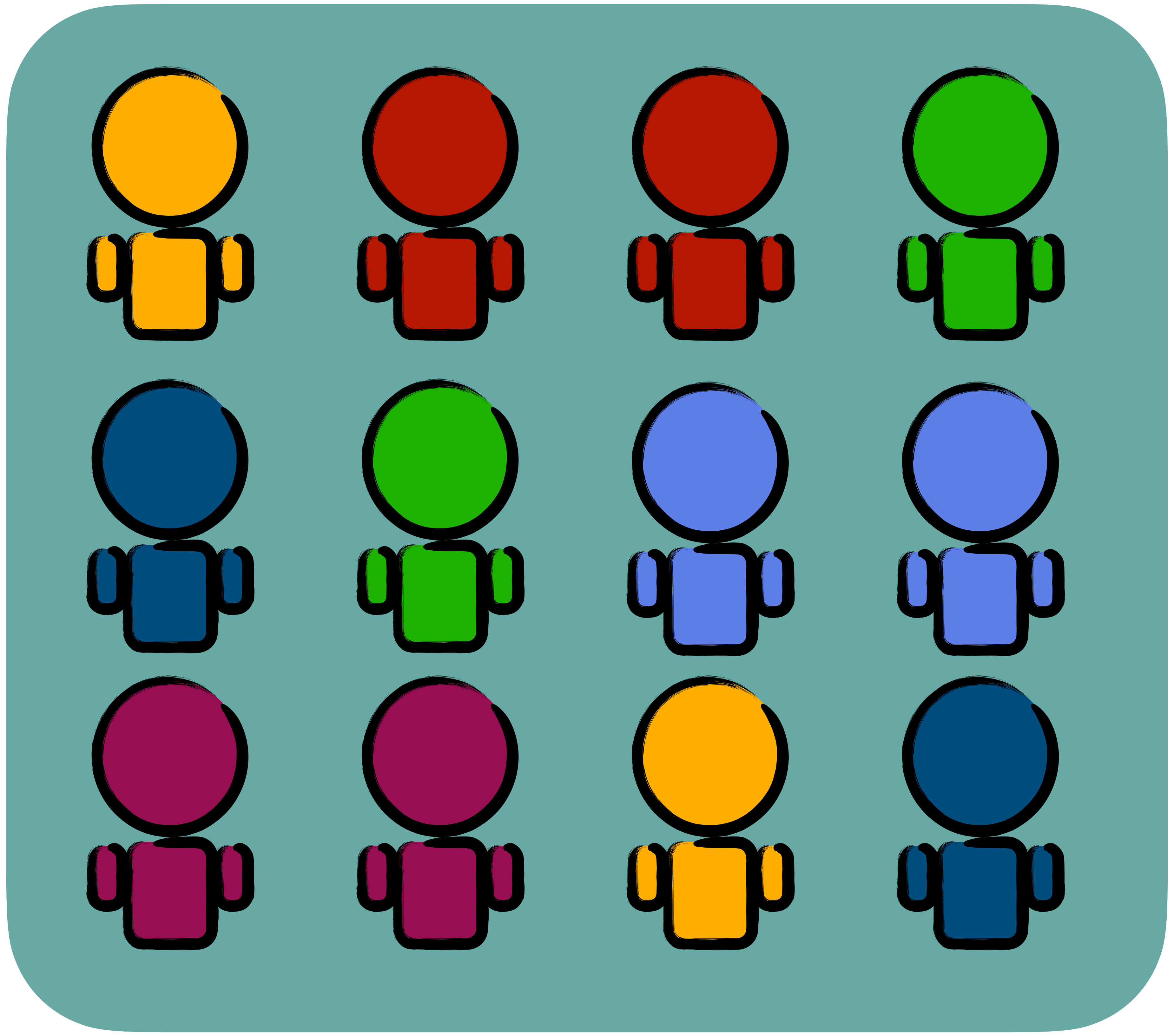
Evolutionary Simulations



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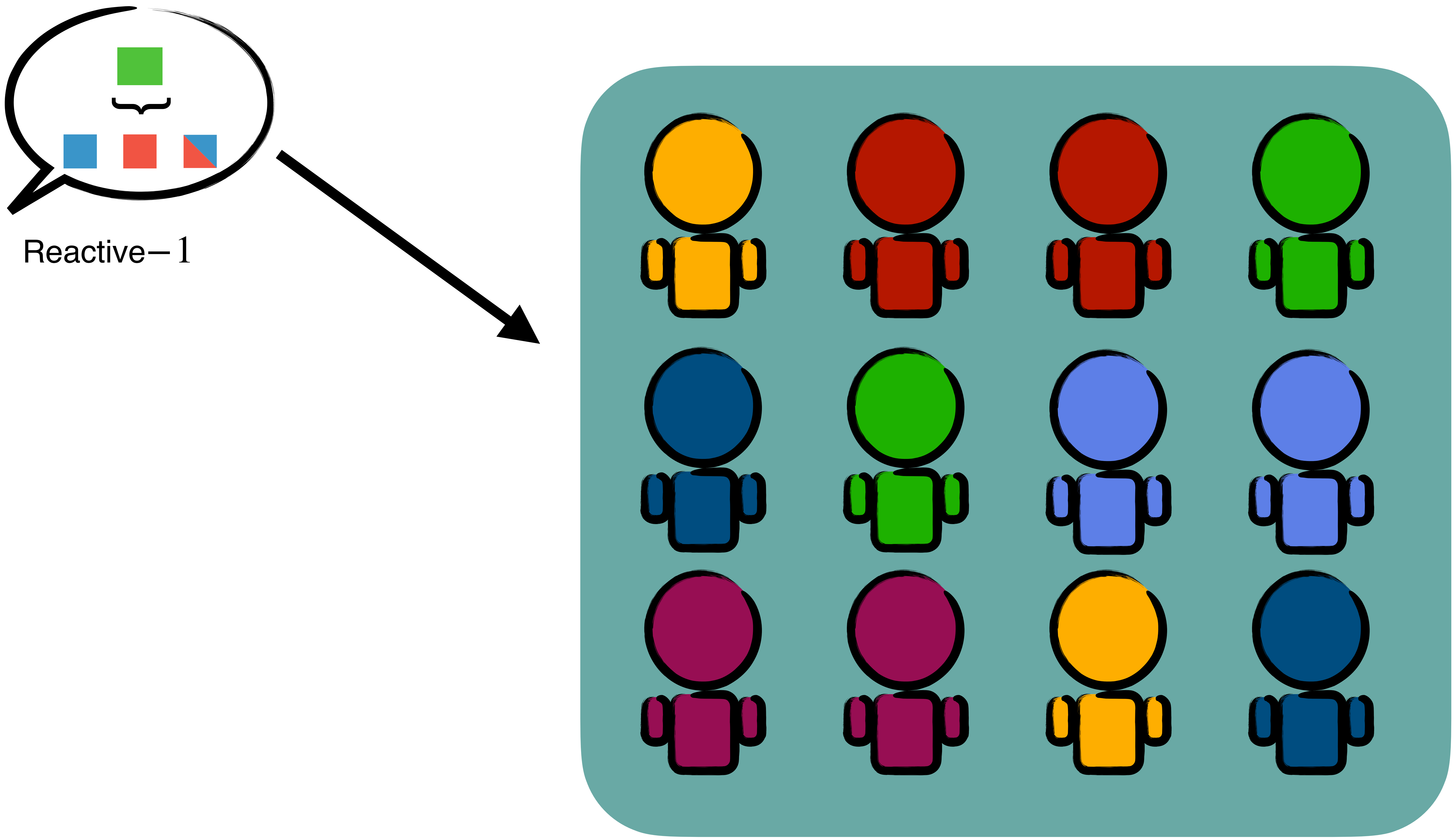

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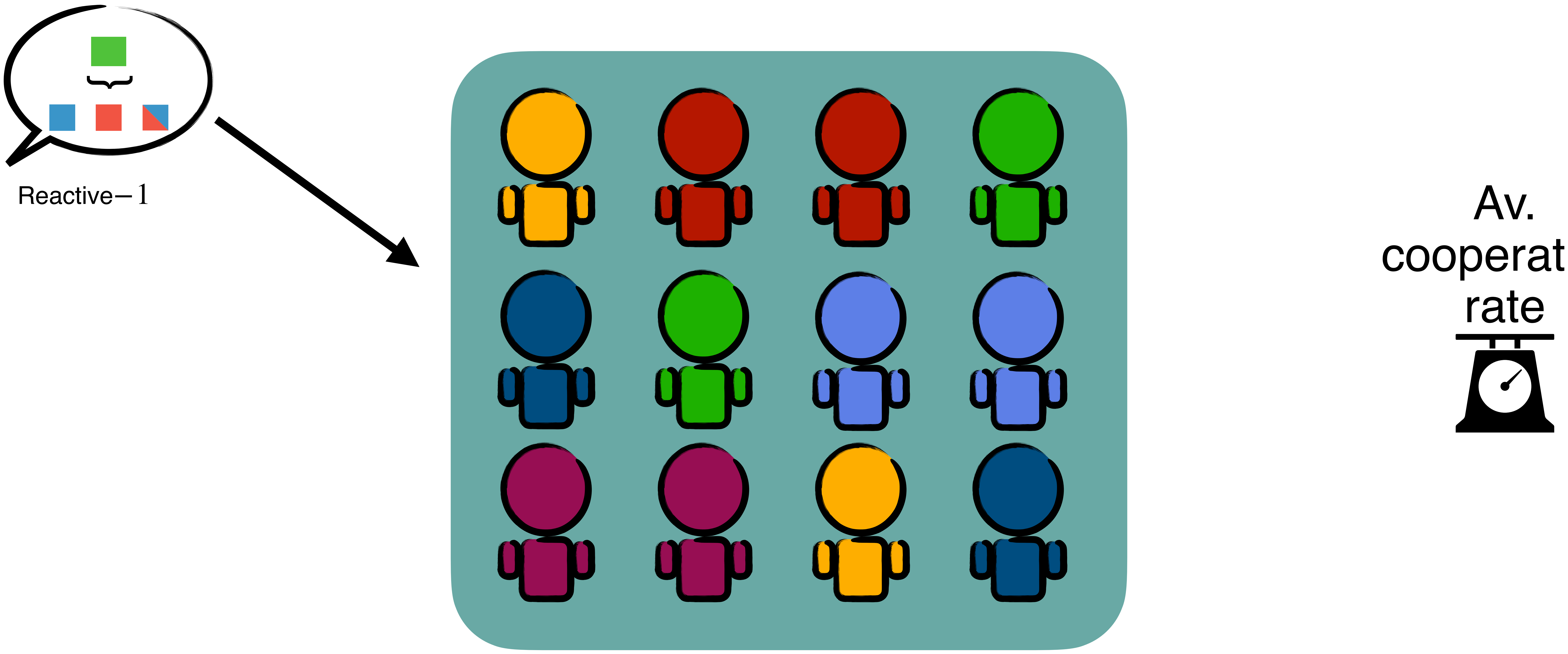

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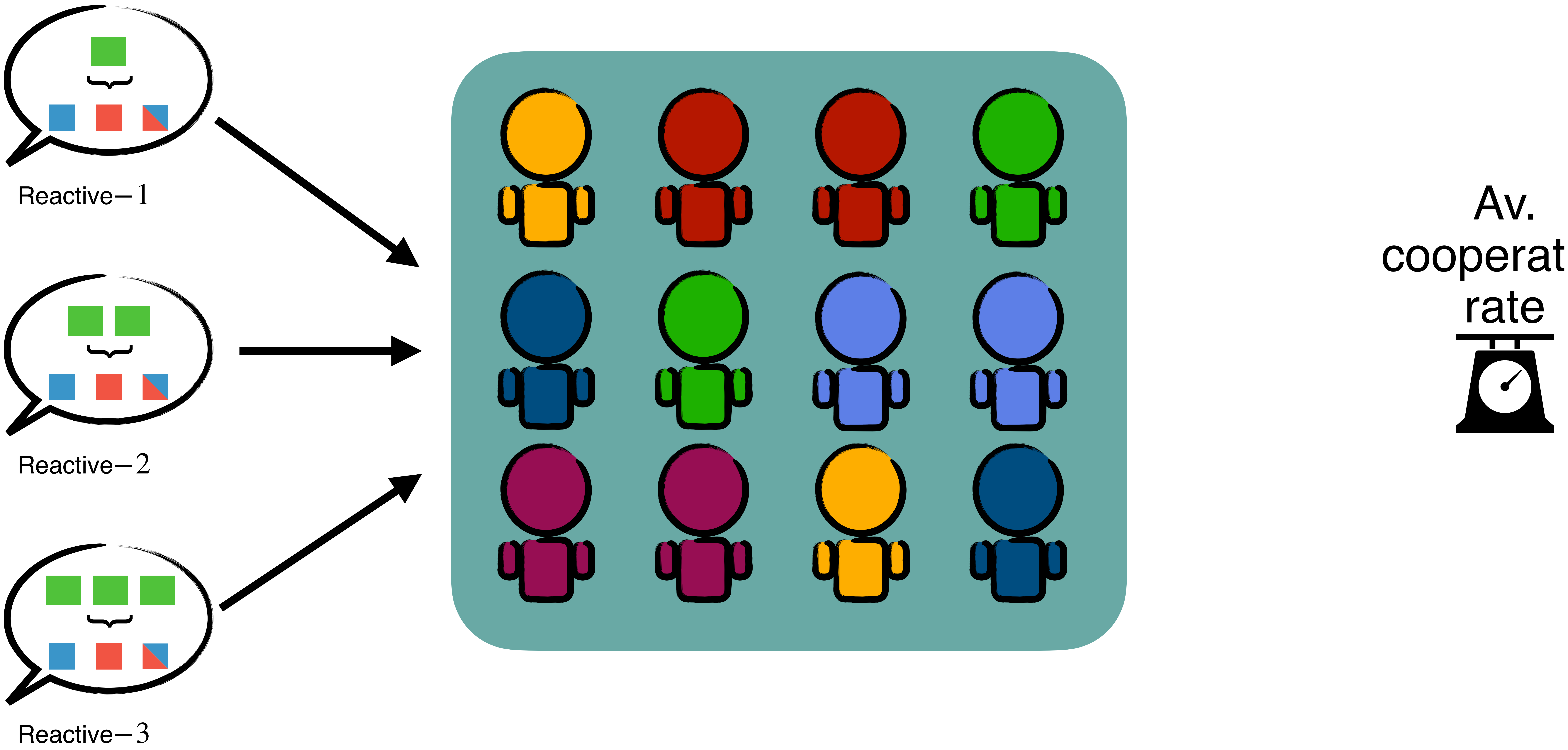

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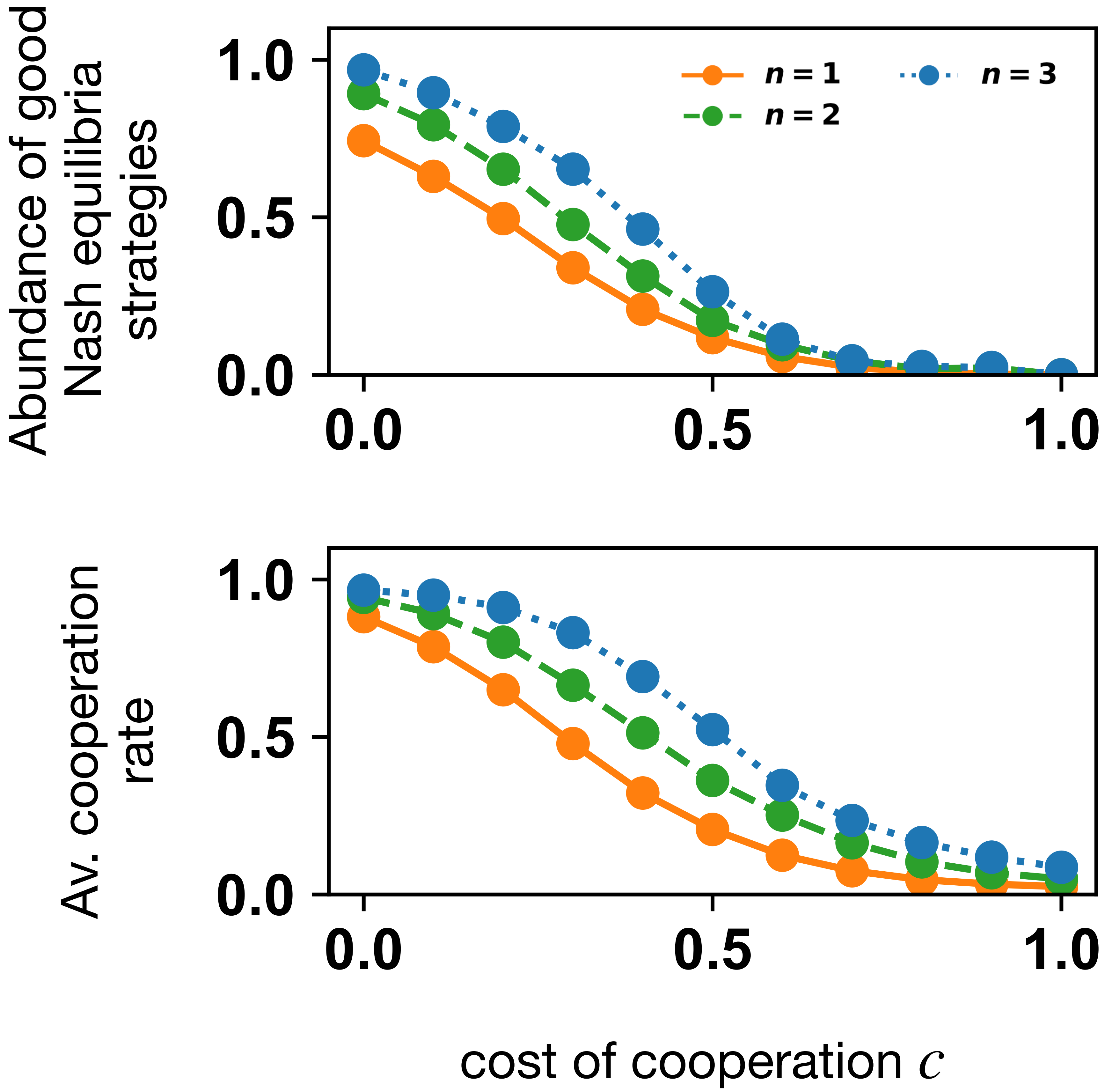
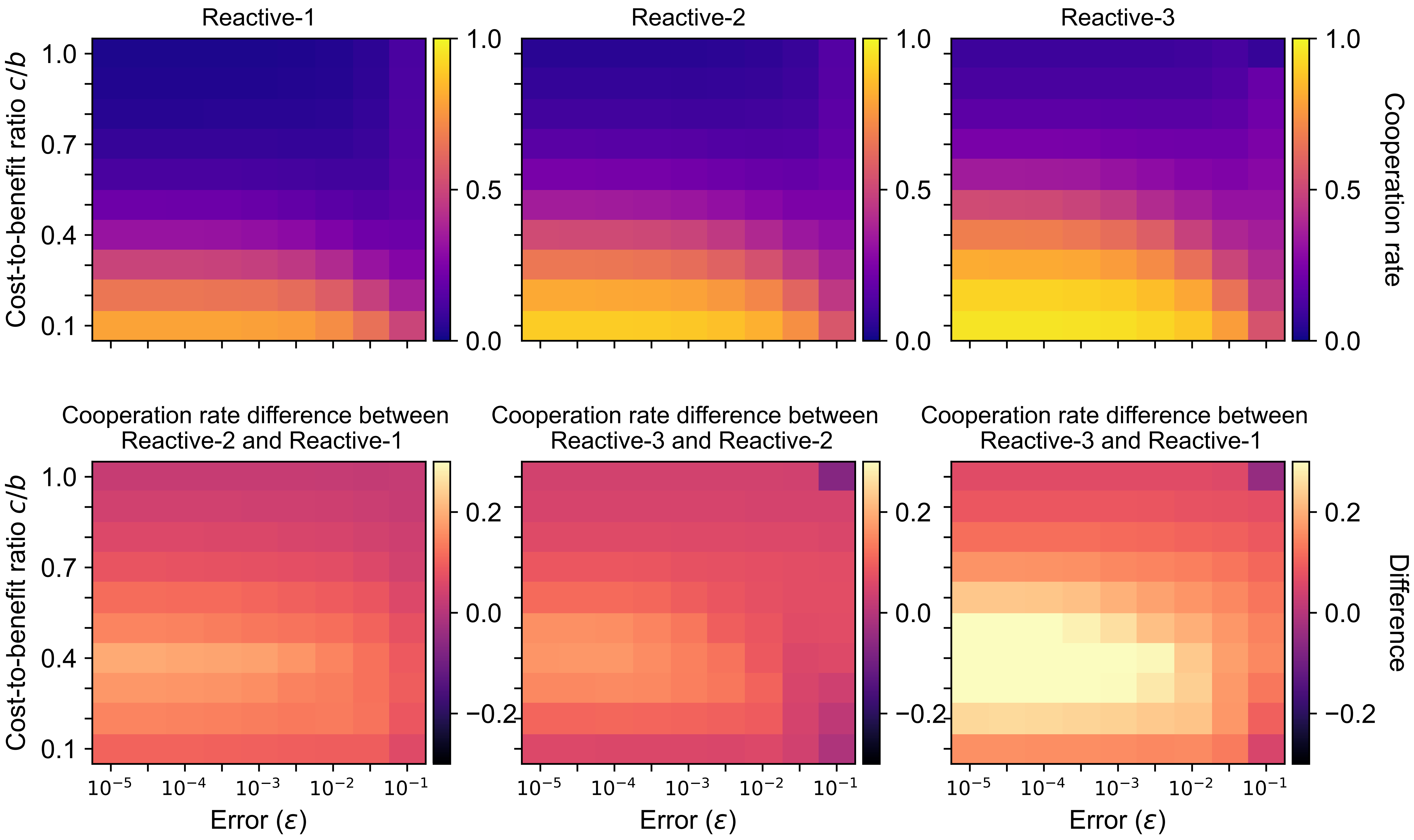


Evolutionary Simulations with Errors



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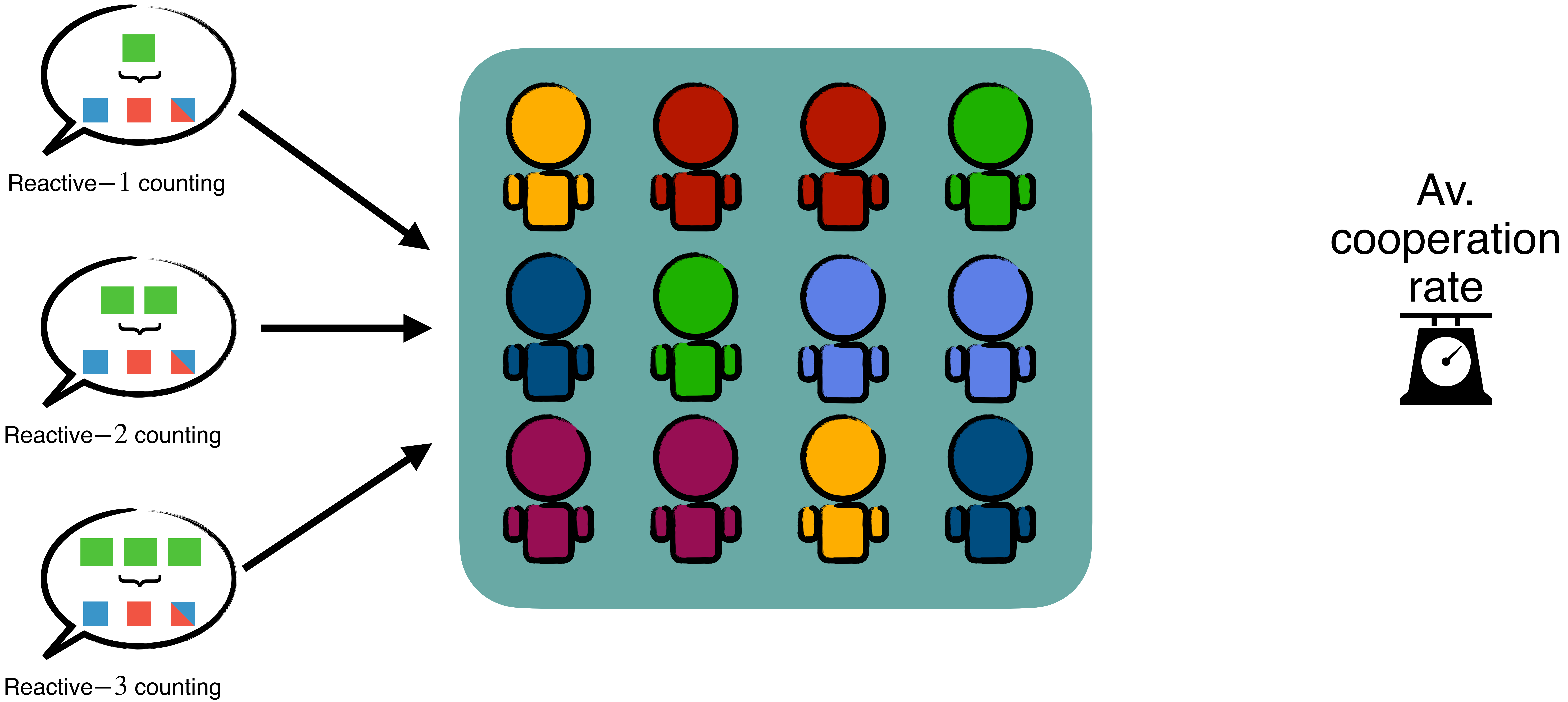

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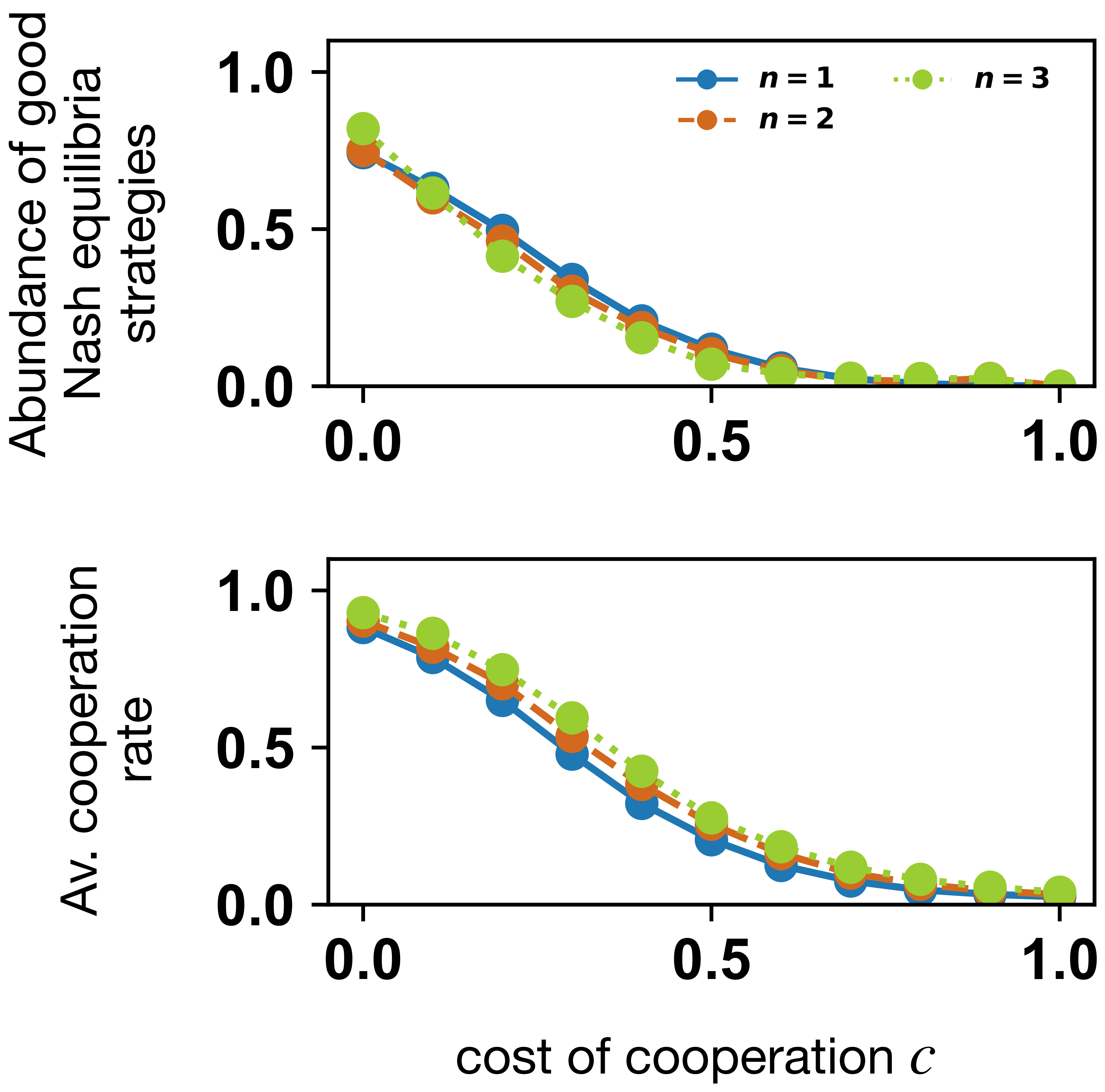

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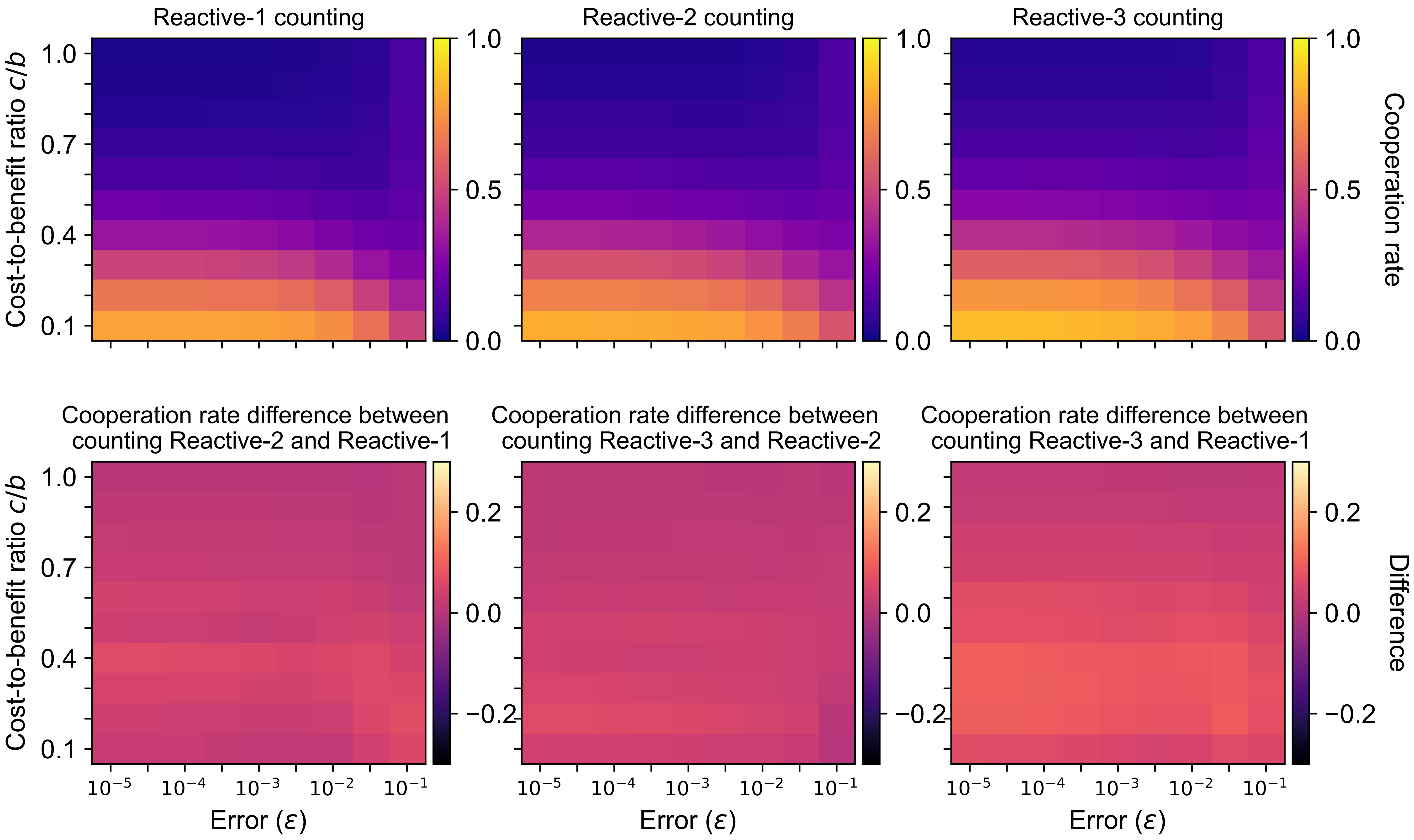

1.

Algorithm to verify whether a given reactive-n strategy is an equilibrium.

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input: p, n
pure_self_reactive_strategies ← { p̃ | p̃ ∈ {0,1}^(2^n) };
isNash ←True;
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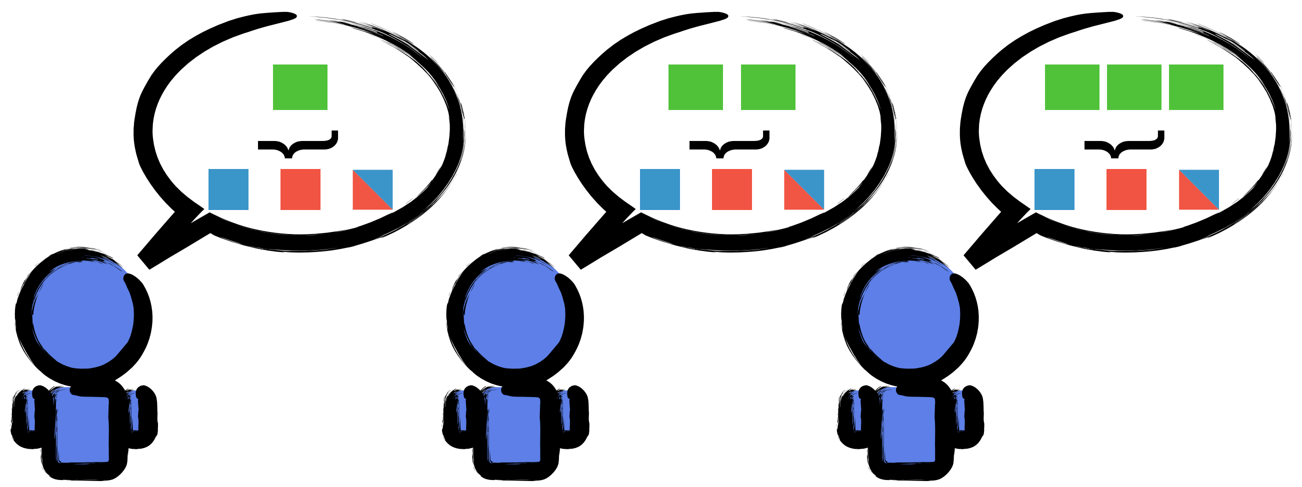
Fully characterize cooperative & defective equilibria for $n = 2$ and $n = 3$.



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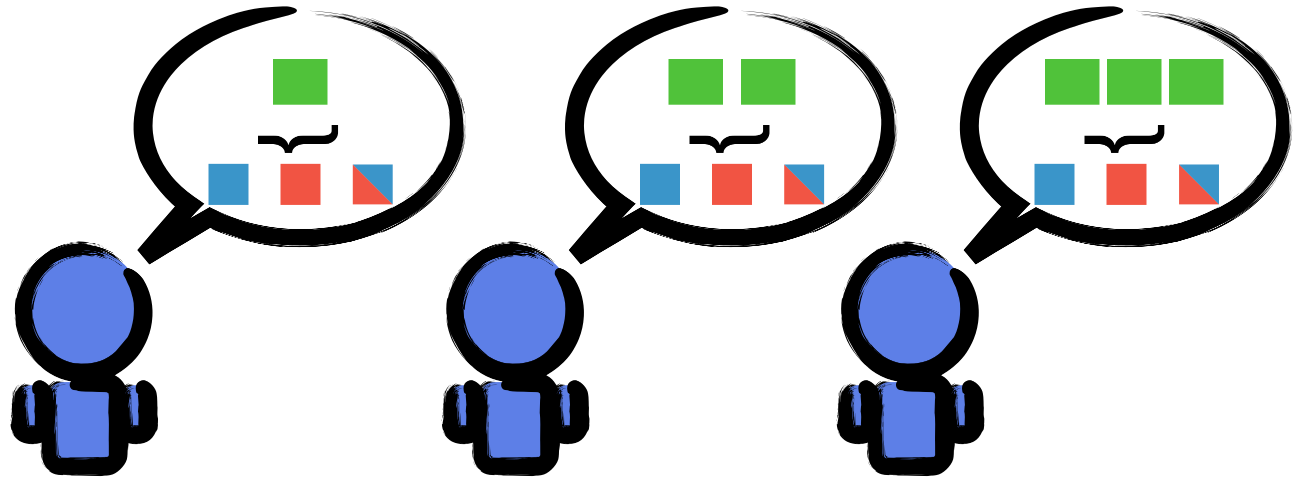

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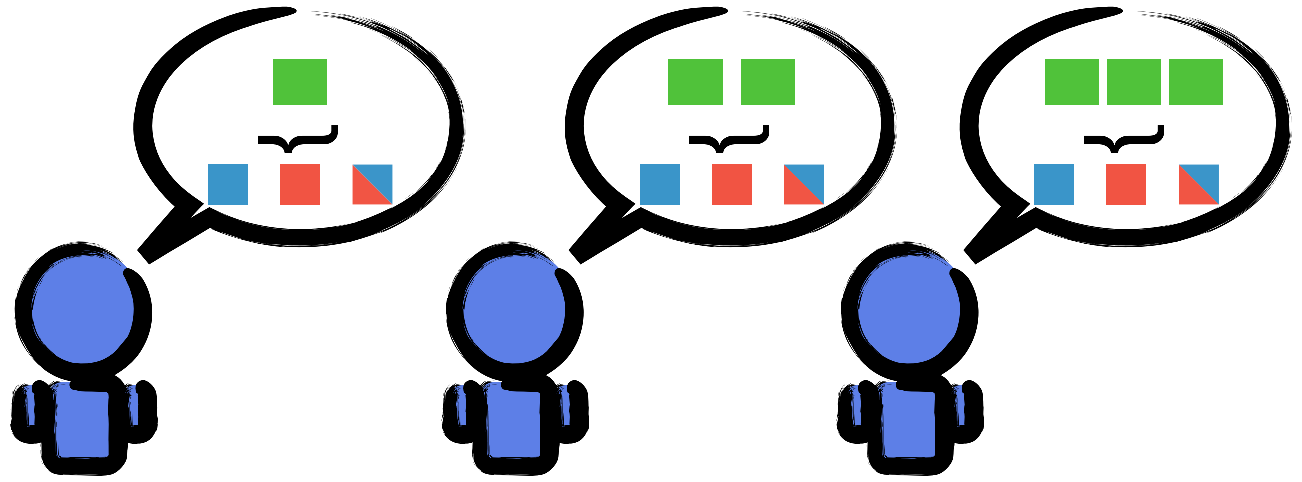

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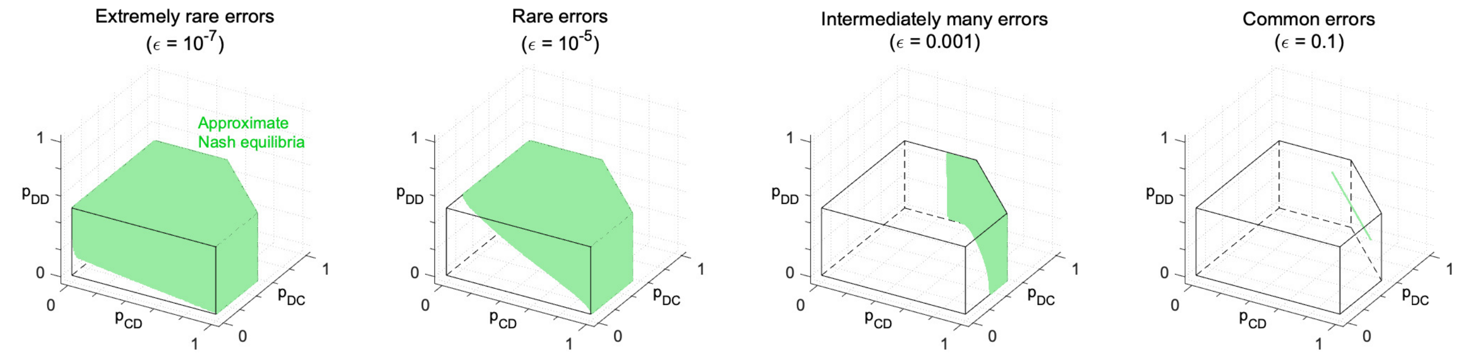
4.

Explore the effects of implementation errors.



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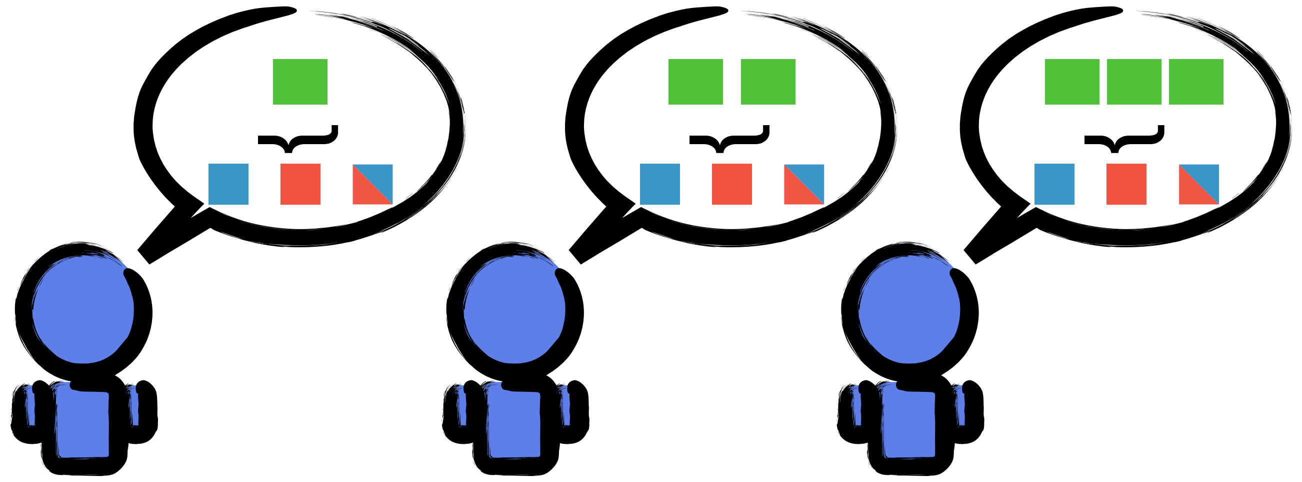
Algorithm to verify whether a given reactive-n strategy is an equilibrium.

```
input: p, n
pure_self_reactive_strategies ← { p̃ | p̃ ∈ {0,1}^(2^n) };
isNash ←True;
for p̃ ∈ pure_self_reactive_strategies do
    if p is not a best response p̃ to then
        isNash ←False;
return (p, isNash);
```

[11] Levínský R., Neyman A., Zelený M., 2020. Should I remember more than you? Best responses to factored strategies.

2.

Fully characterize cooperative & defective equilibria for $n = 2$ and $n = 3$.



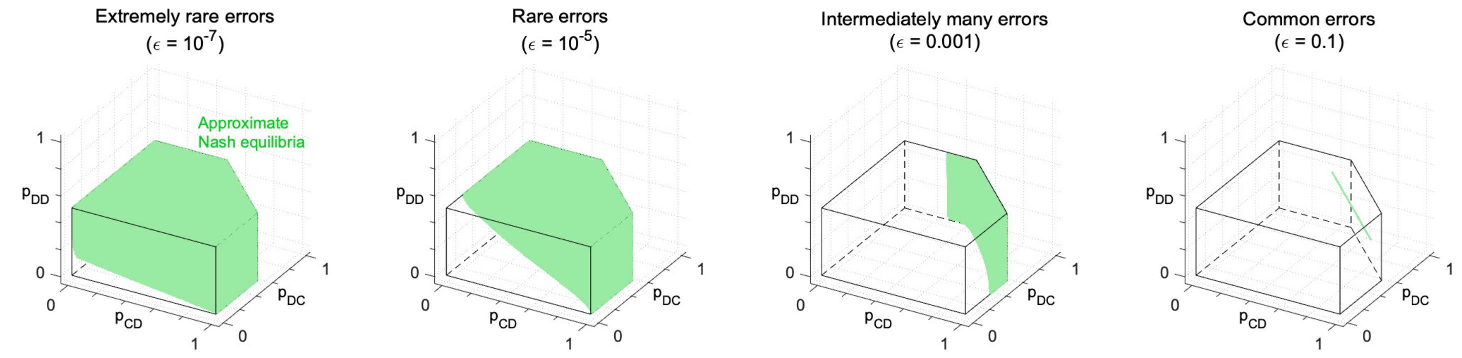
3.

Fully characterize cooperative & defective equilibria for any n for reactive counting strategies.

$$r_{n-k} \leq 1 - \frac{k}{n} \cdot \frac{c}{b} \quad \text{for } k \in \{1,2,\ldots,n\}\,.$$

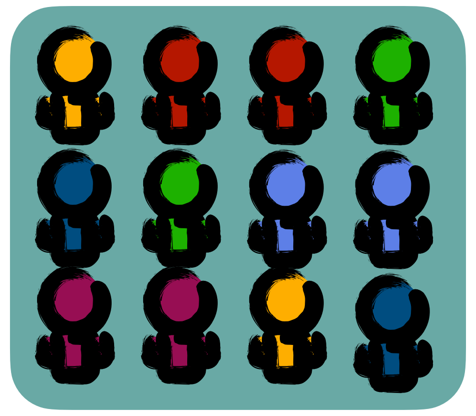
4.

Explore the effects of implementation errors.



5.

Performed evolutionary simulations varying several key parameters.



1.

Algorithm to verify whether a given reactive-n strategy is an equilibrium.

input: p, n

pure_self_reactive_strategies ← $\left\{ \tilde{\mathbf{p}} \mid \tilde{\mathbf{p}} \in \{0,1\}^{2^n} \right\}$;

isNash ←True;

for $\tilde{\mathbf{p}} \in$ *pure_self_reactive_strategies* **do**

  if p is not a best response $\tilde{\mathbf{p}}$ to **then**

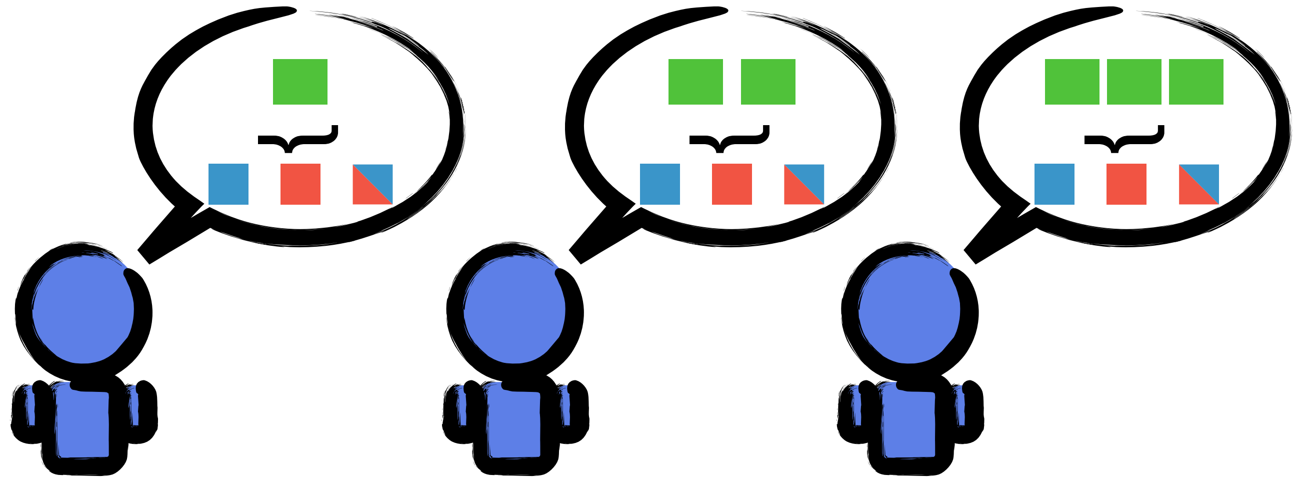
    isNash ←False;

return (**p**, isNash);

[11] Levínský R., Neyman A., Zelený M., 2020. Should I remember more than you? Best responses to factored strategies.

2.

Fully characterize cooperative & defective equilibria for $n = 2$ and $n = 3$.



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Fully characterize cooperative & defective equilibria for any n for reactive counting strategies.

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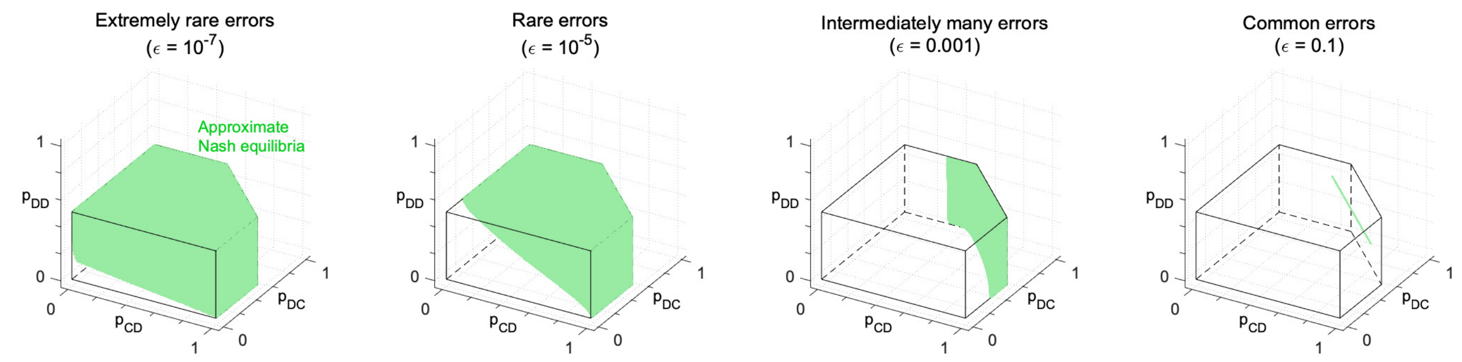
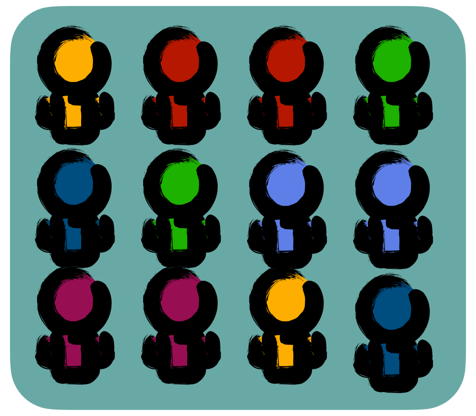
Explore the effects of implementation errors.



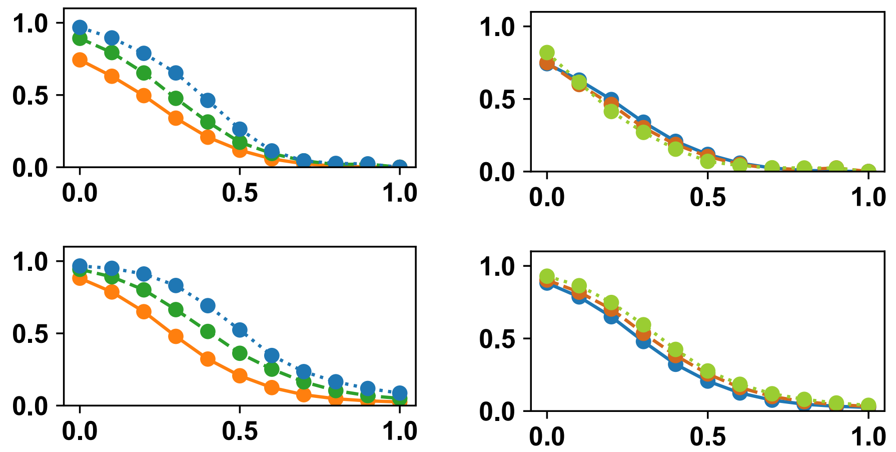
5.

Performed evolutionary simulations varying several key parameters.



6.

Longer memory helps sustain cooperation.



1. Introduction and motivation

2. Conditional cooperation with longer memory

Published *PNAS:* https://doi.org/10.1073/pnas.2420125121



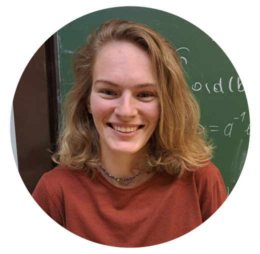
3. Complete strategy spaces of direct reciprocity

Under review
PNAS



Can I afford to remember less than you?

Under review
Economics Letters



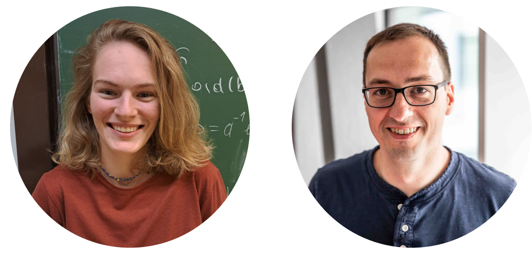

1. Introduction and motivation

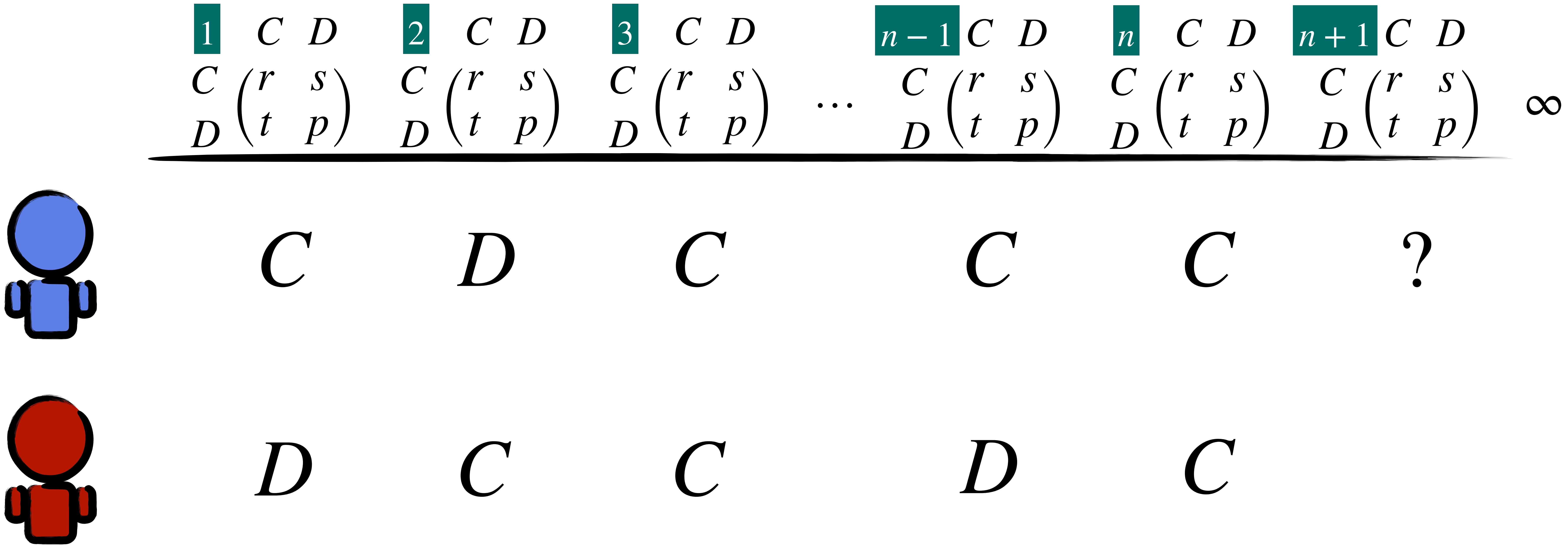
2. Conditional cooperation with longer memory

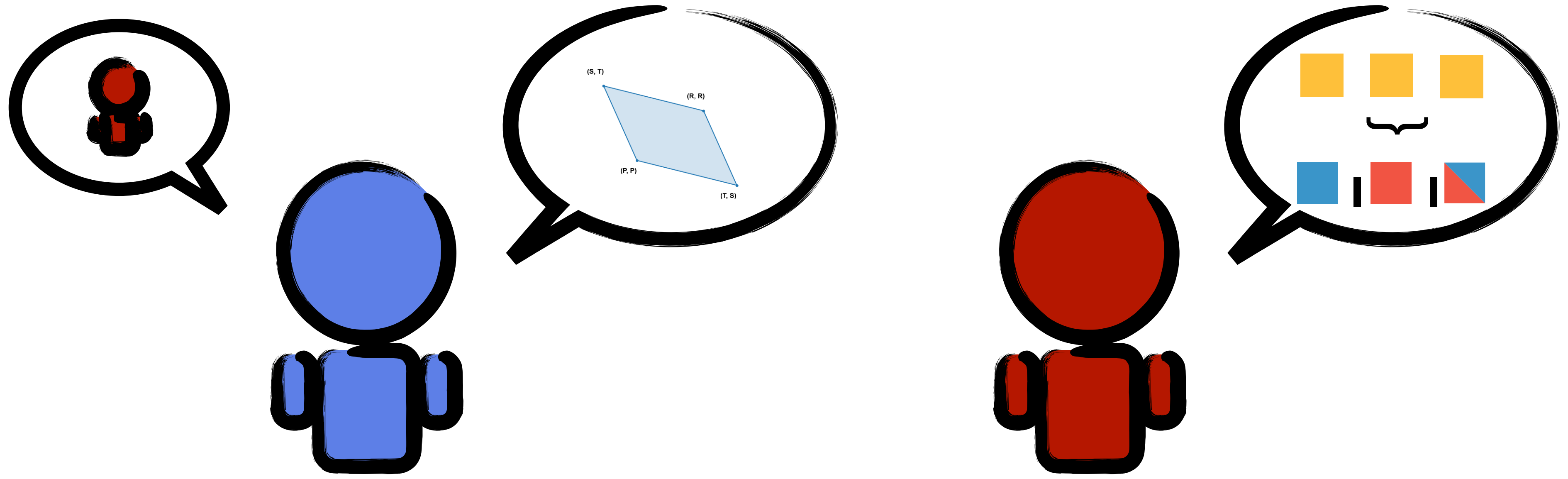
Published *PNAS:* https://doi.org/10.1073/pnas.2420125121

3.

Complete strategy spaces of direct reciprocity	Can I afford to remember less than you?
Under review *PNAS*	Under review *Economics Letters*

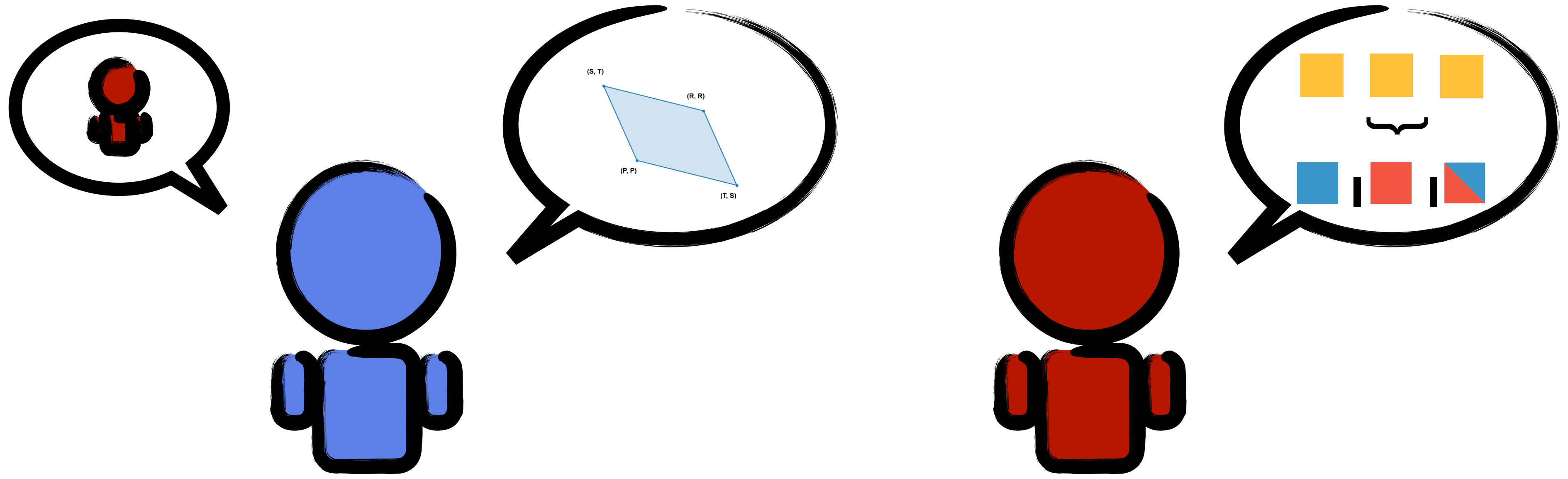



1
2
3
n − 1
n
n + 1
C D
C (r s)
D (t p)
…
∞
C D C C C ?
D C C D C



[1] Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent.

[12] Glynatsi N.E., Akin E., Nowak M.A., Hilbe C. 2024. Conditional strategies with longer memory.



Memory-n

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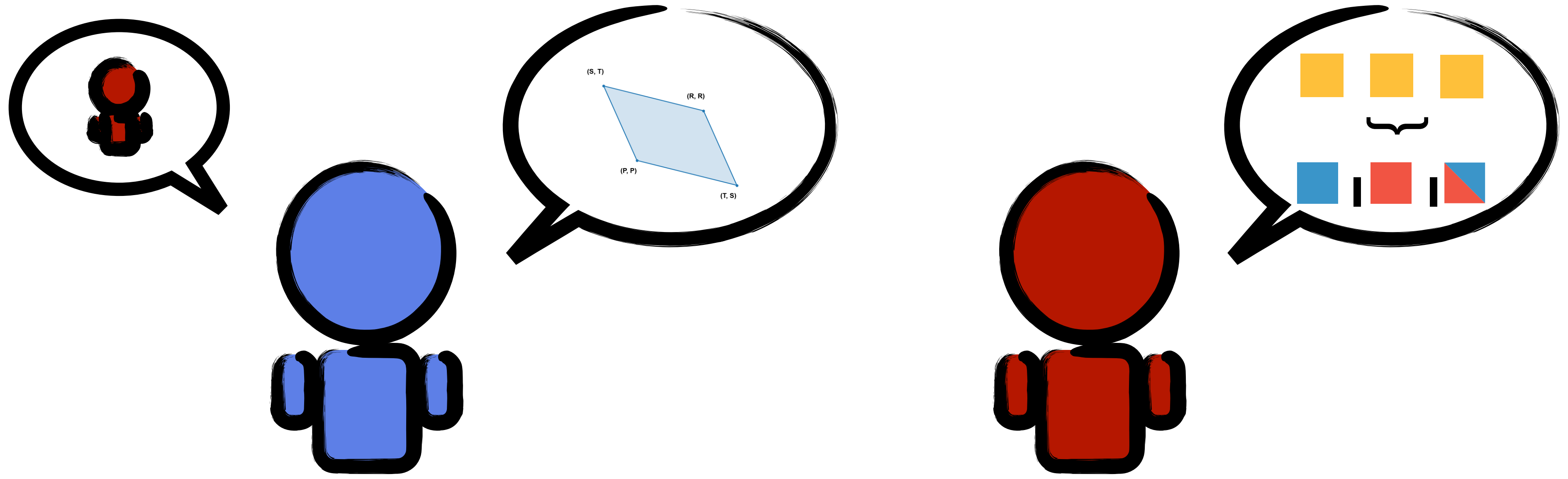

Memory-n [1]

Memory-n

[1] Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent.

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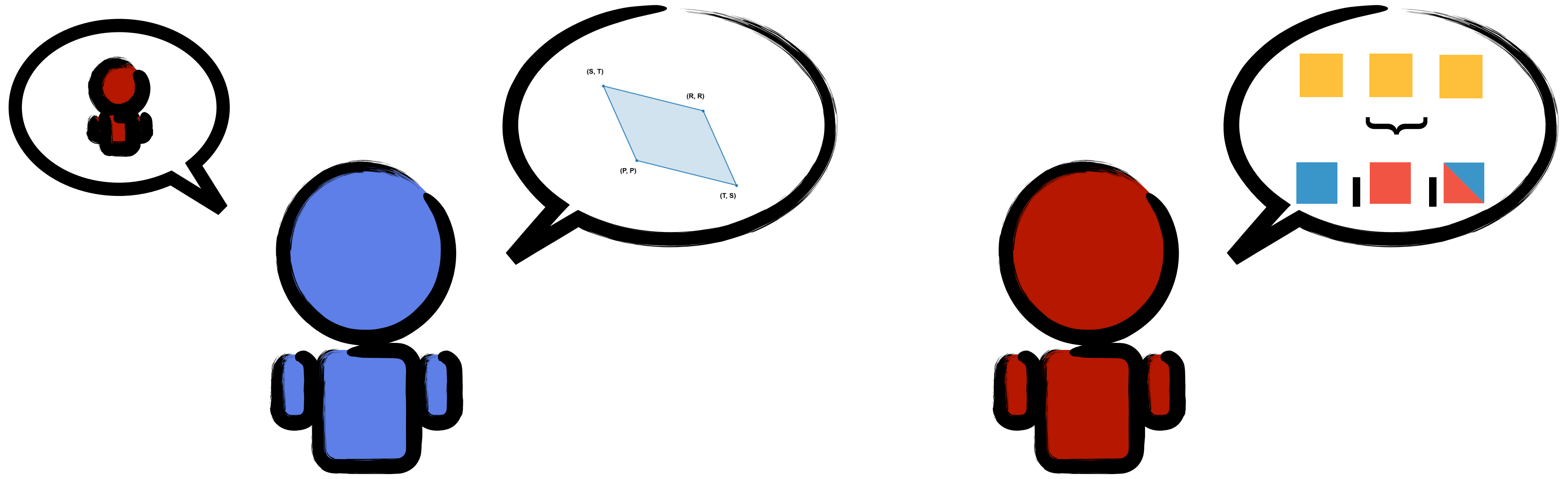


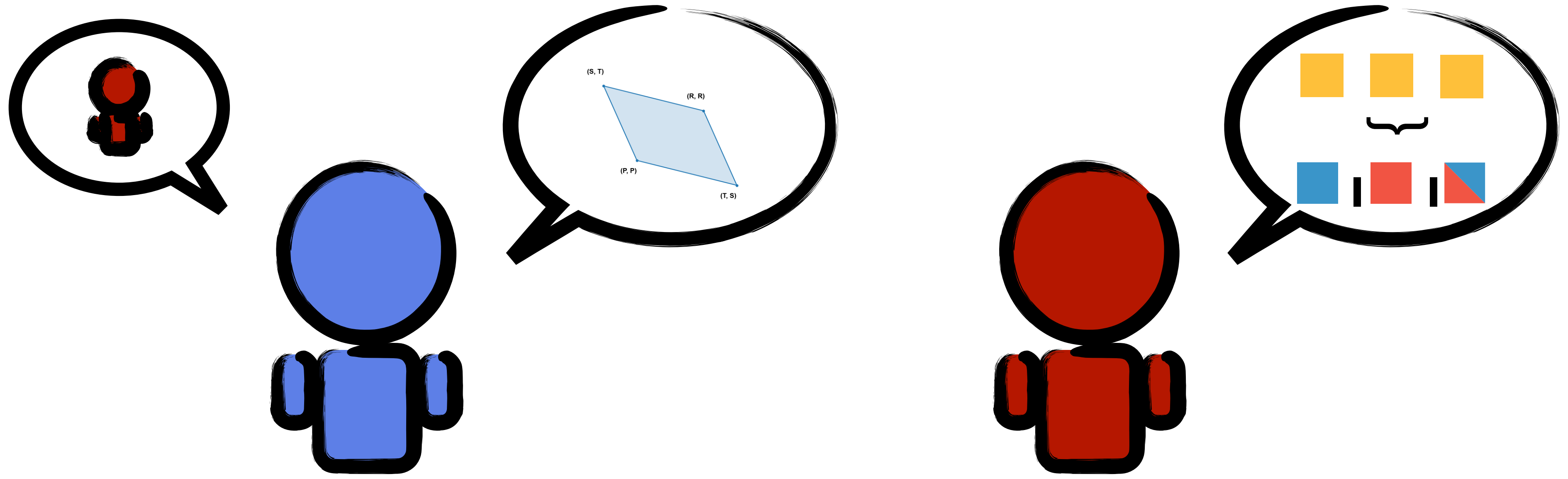
Memory-n [1]

Memory-n

Reactive-n

[1] Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent.

[12] Glynatsi N.E., Akin E., Nowak M.A., Hilbe C. 2024. Conditional strategies with longer memory.



Memory-n [1] Memory-n

Memory-n [1] Reactive-n

[1] Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent.

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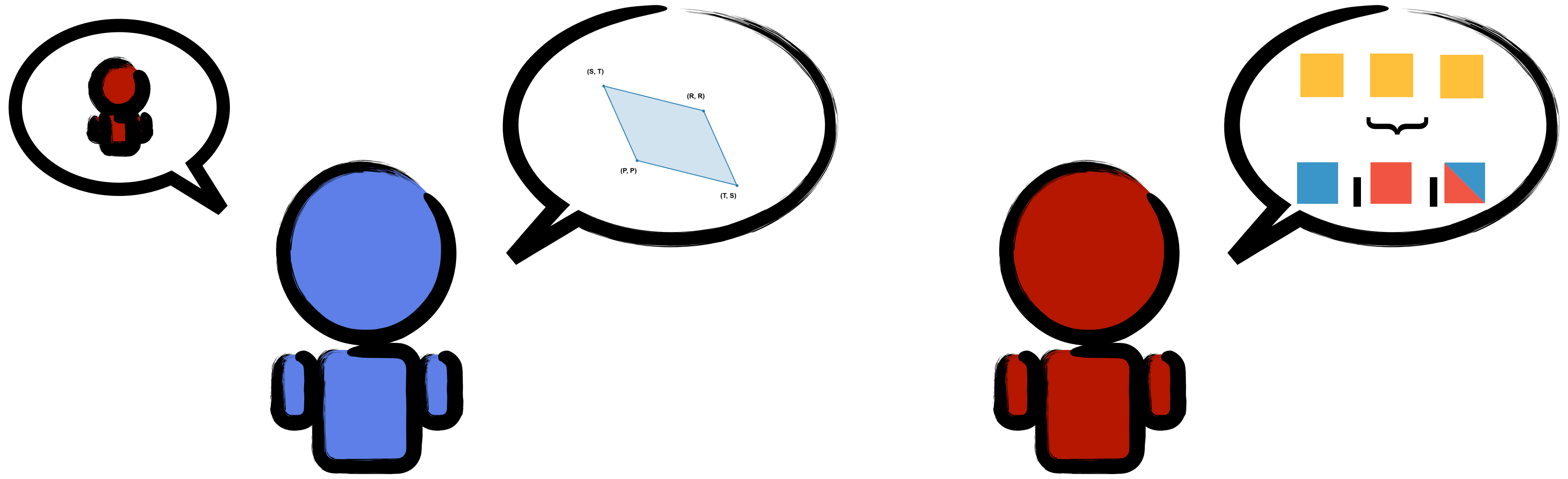

Memory-n [1] — Memory-n

Memory-n [1] — Reactive-n

Self reactive-n [12] — Reactive-n

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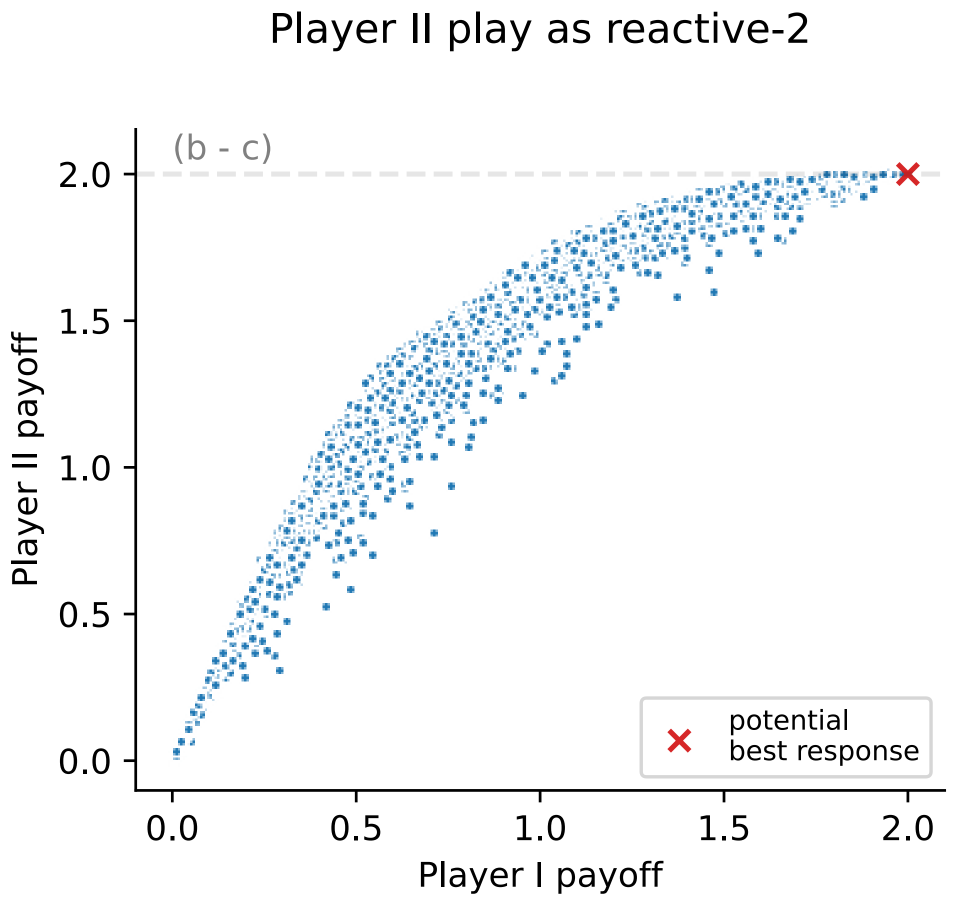
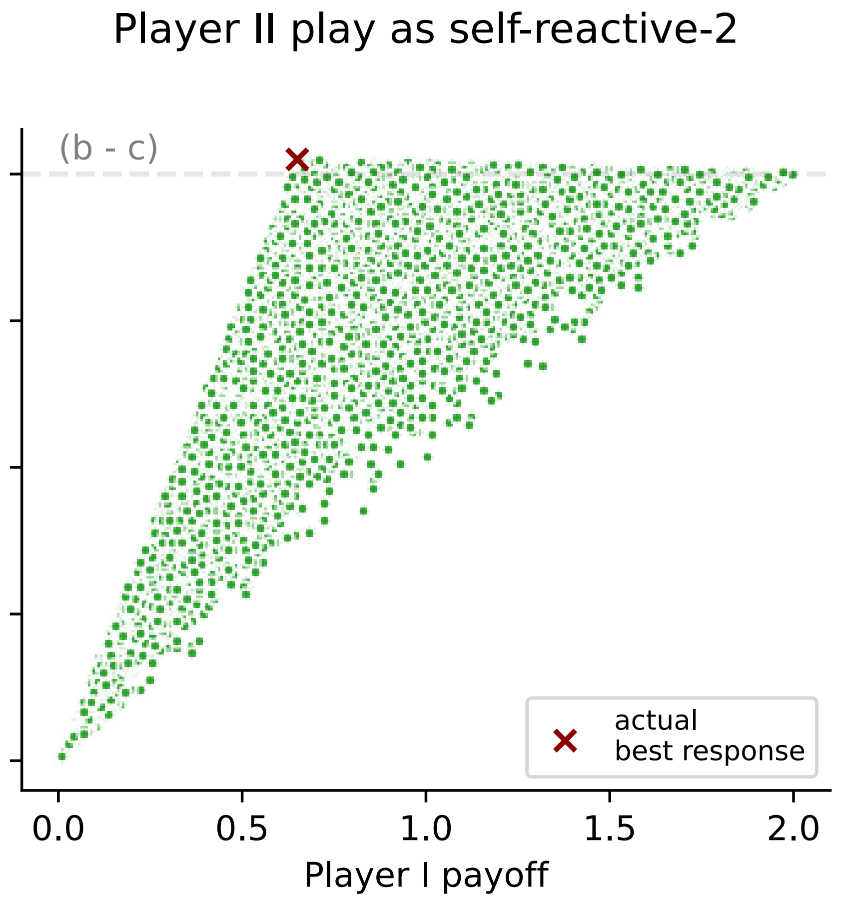
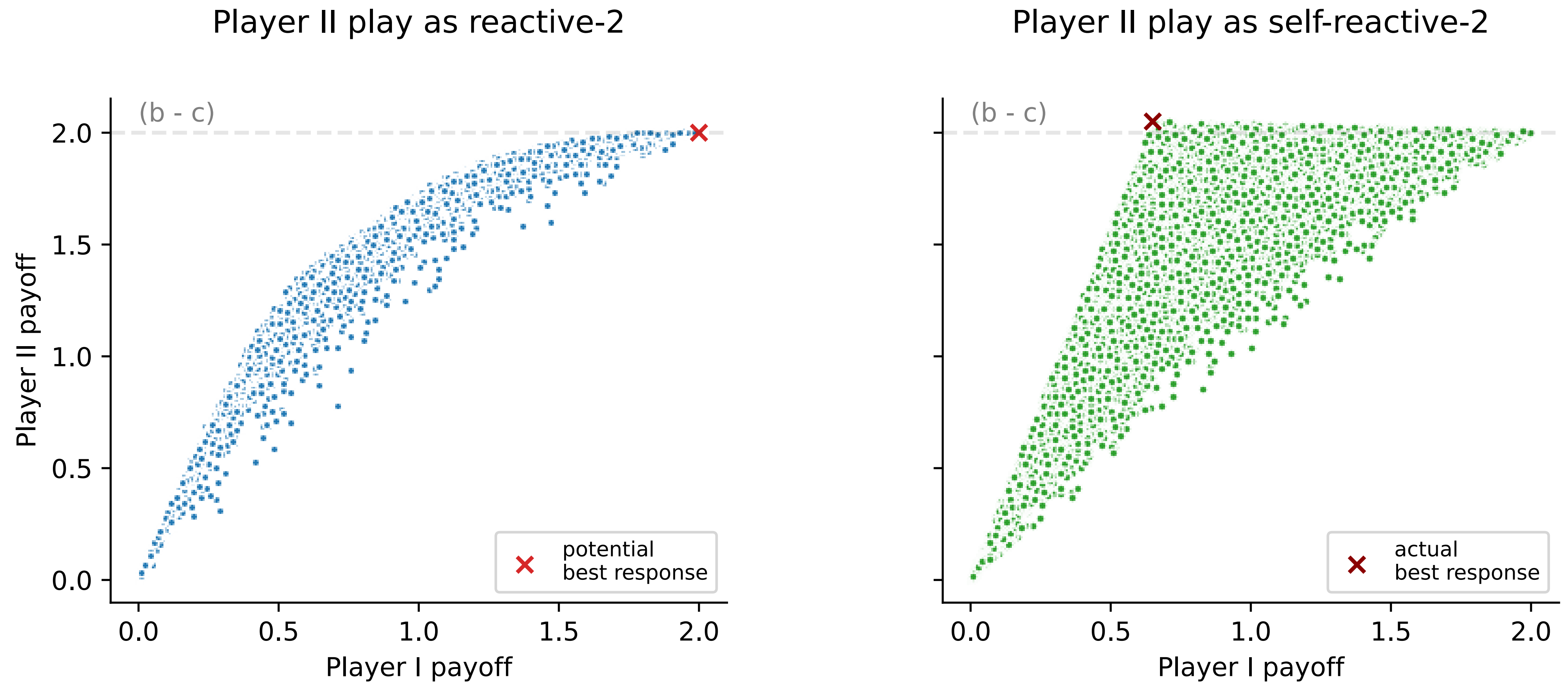
Complete strategy spaces

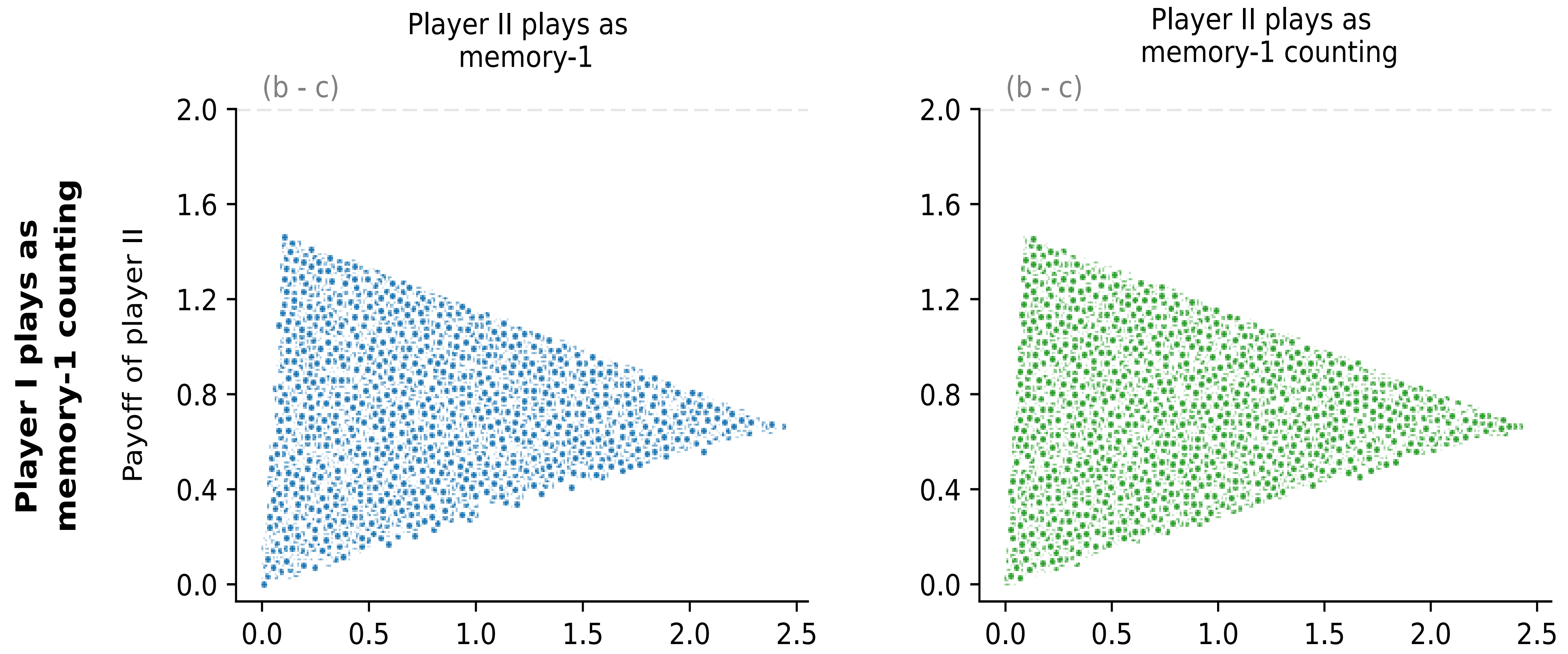
Complete strategy spaces



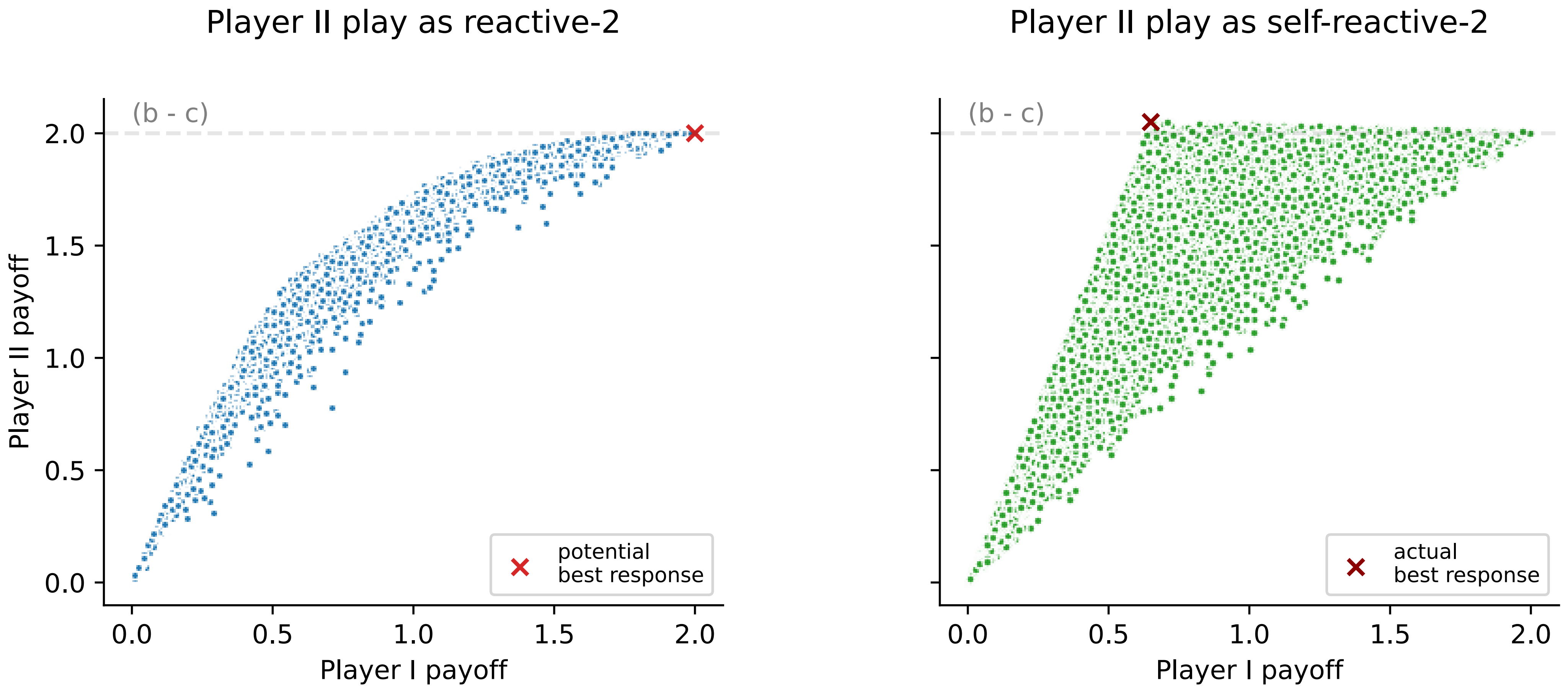


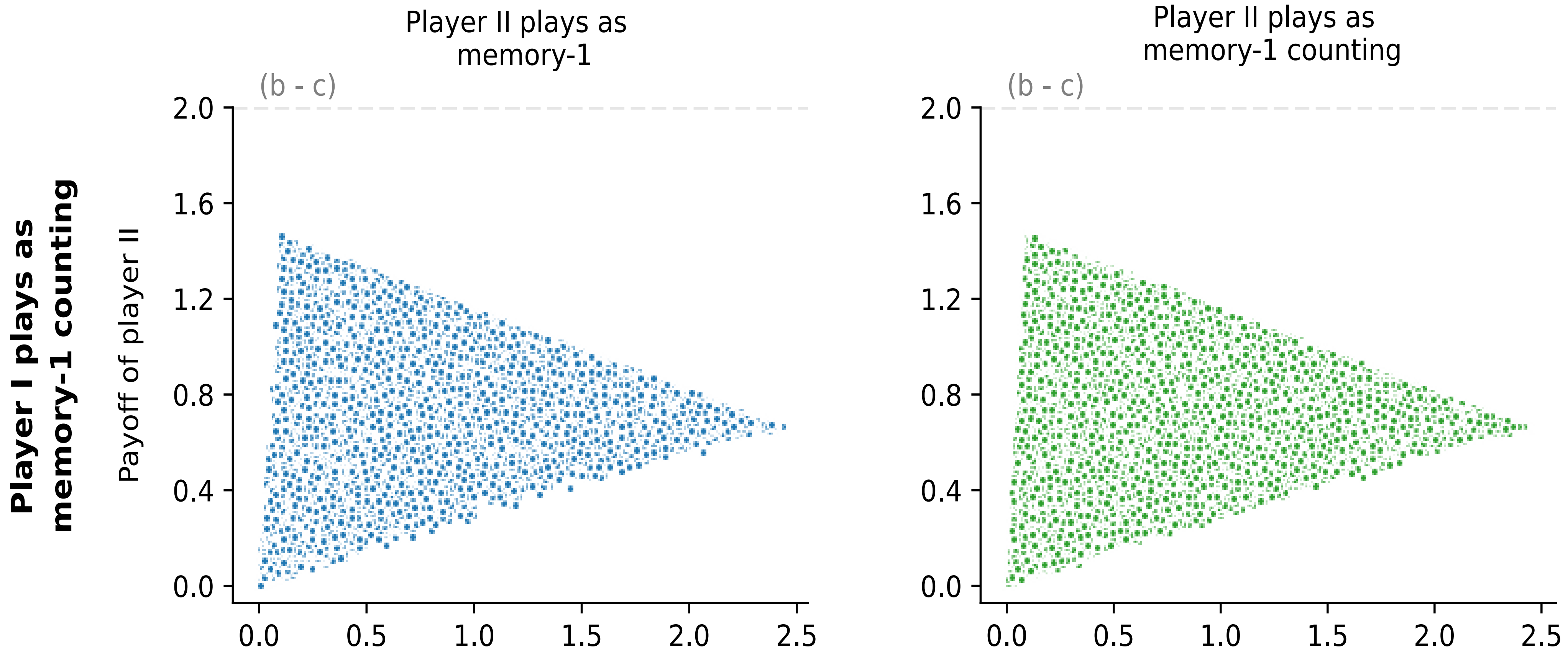
Complete strategy spaces

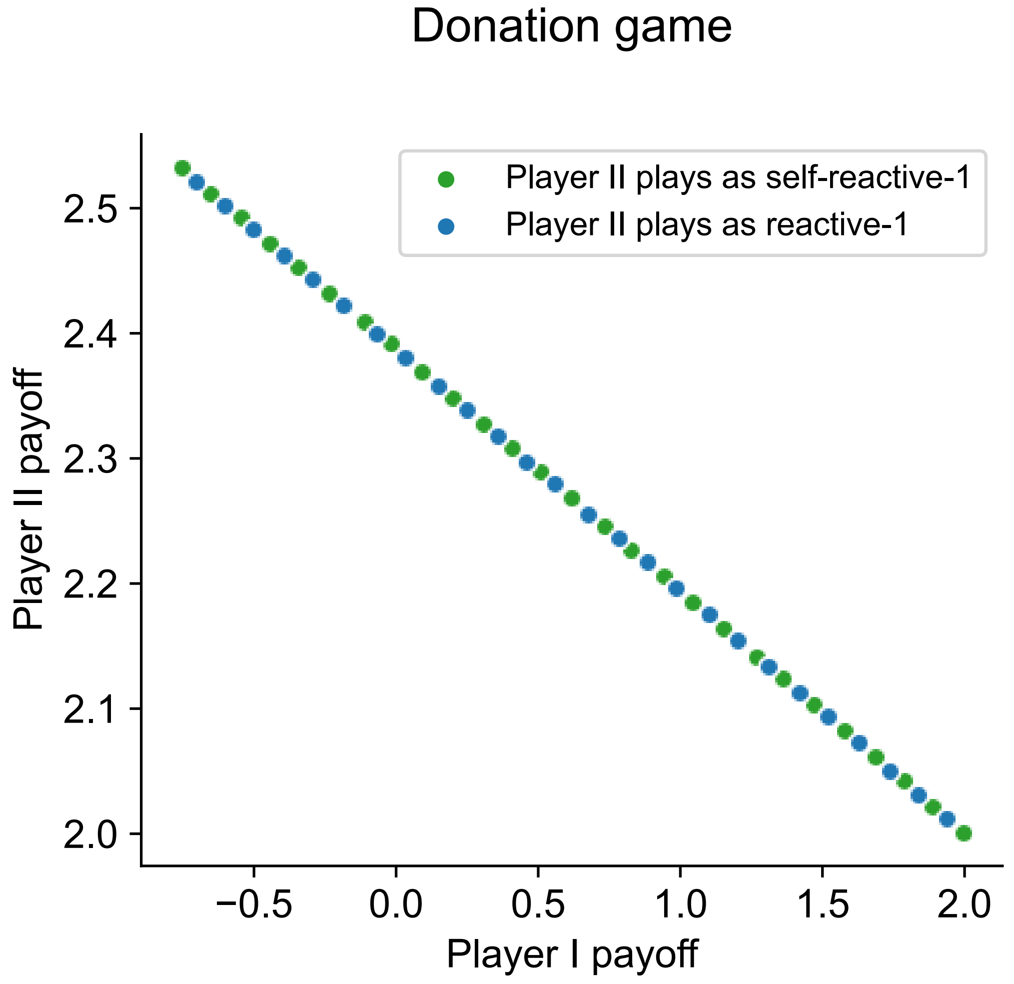


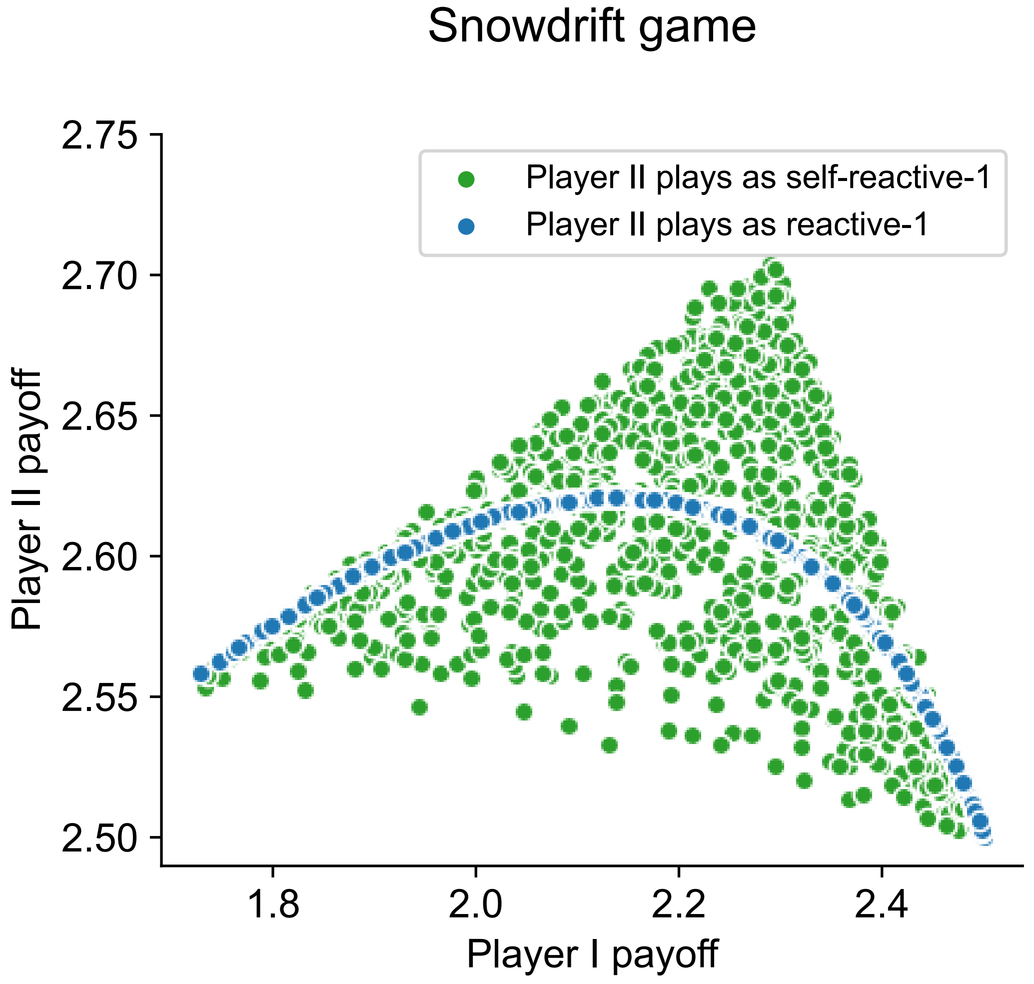


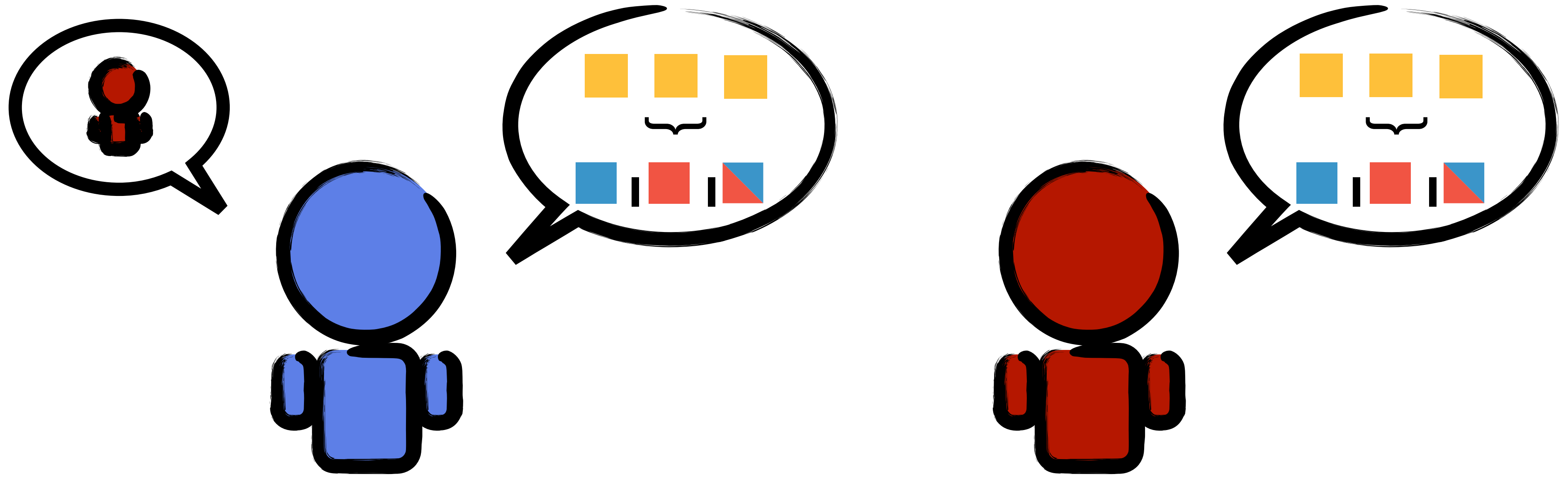
Complete strategy spaces











Memory-n [1] | Memory-n

Memory-n [1] | Reactive-n

Self reactive-n [11], [12] | Reactive-n

[1] Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent.

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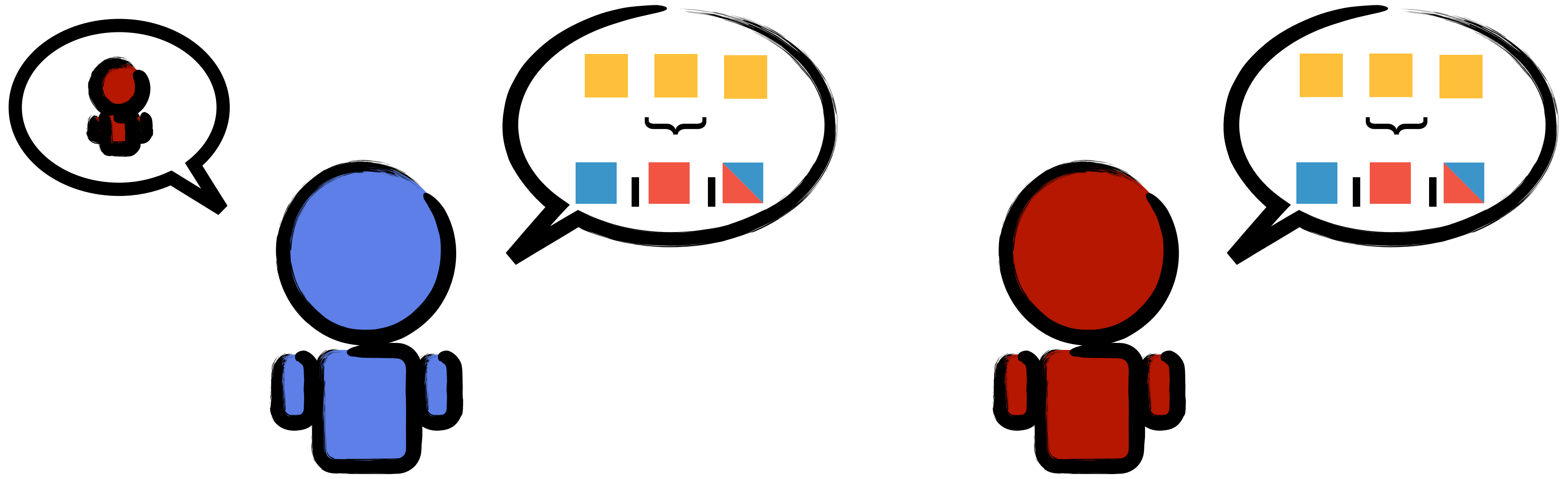

Memory-n [1] | Memory-n

Pure memory-n [11] | Memory-n

Memory-n [1] | Reactive-n

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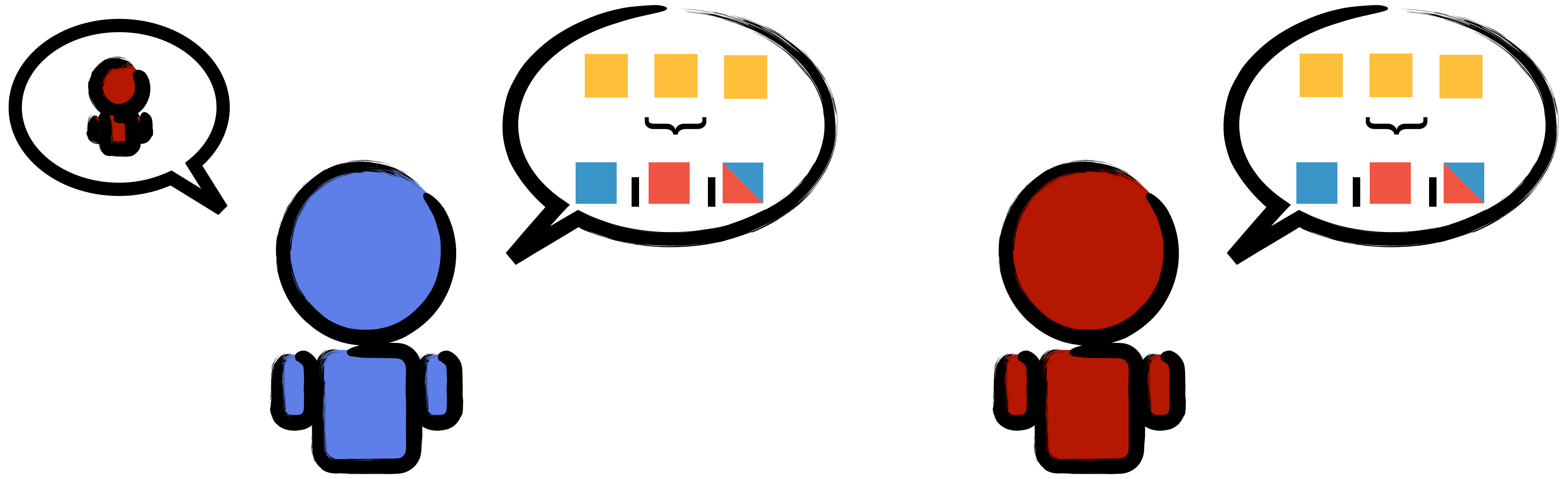

Memory-n [1]	Memory-n
Pure memory-n [11]	Memory-n

Memory-n [1]	Reactive-n
Self reactive-n [11], [12]	Reactive-n
Pure self reactive-n [11], [12]	Reactive-n

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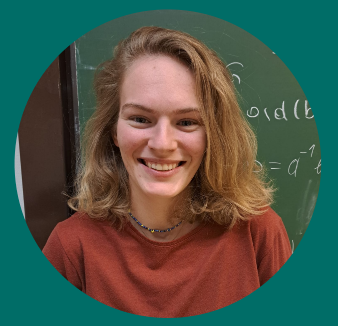
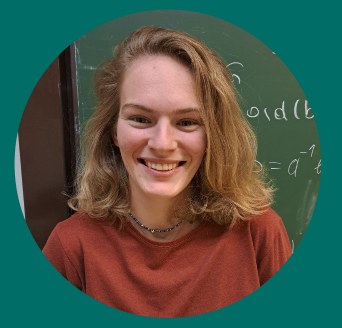
Should I remember more than you?

Can I afford to remember less than you? 

Should I remember more than you?

No ✗

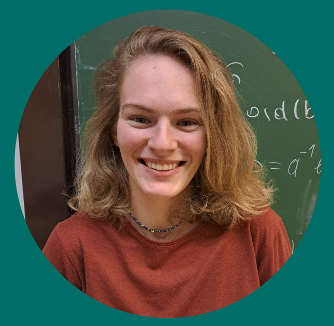
Can I afford to remember less than you?



Should I remember more than you?

No ✗

Can I afford to remember less than you?

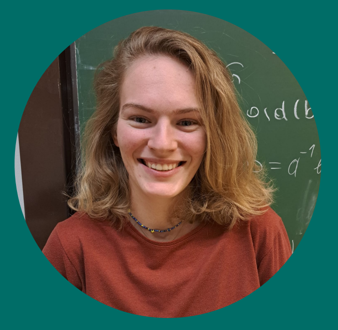


Yes ✓

Should I remember more than you?

No ✗

Can I afford to remember less than you?



Yes ✓

1. The game is additive. $$\begin{array}{c} \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array}$$

$b > c > 0$

2. The opponent follows a reactive-n strategy.

Theorem. Let $\mathbf{p} \in \mathscr{R}_n$ (be a reactive-n strategy) and the game be additive. Then there exists $\tilde{\mathbf{p}}$ in the set of pure self-reactive-$(n-1)$ strategies that is a best response.

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n	1	2	3	4	5
pure memory-n					
pure self-reactive-n					
pure self-reactive-$(n-1)$					

Number of equations to be checked.

Theorem. Let $\mathbf{p} \in \mathscr{R}_n$ (be a reactive-n strategy) and the game be additive. Then there exists $\tilde{\mathbf{p}}$ in the set of pure self-reactive-$(n-1)$ strategies that is a best response.

n	1	2	3	4	5
pure memory-n	16				
pure self-reactive-n	4				
pure self-reactive-$(n-1)$	2				

Number of equations to be checked.

Theorem. Let $\mathbf{p} \in \mathscr{R}_n$ (be a reactive-n strategy) and the game be additive. Then there exists $\tilde{\mathbf{p}}$ in the set of pure self-reactive-$(n-1)$ strategies that is a best response.

n	1	2	3	4	5
pure memory-n	16	65,536			
pure self-reactive-n	4	16			
pure self-reactive-$(n-1)$	2	4			

Number of equations to be checked.

Theorem. Let $\mathbf{p} \in \mathscr{R}_n$ (be a reactive-n strategy) and the game be additive. Then there exists $\tilde{\mathbf{p}}$ in the set of pure self-reactive-$(n-1)$ strategies that is a best response.

n	1	2	3	4	5
pure memory-n	16	65,536	1,844,674,407,370,955,161		
pure self-reactive-n	4	16	256		
pure self-reactive-$(n-1)$	2	4	16		

Number of equations to be checked.

Theorem. Let $\mathbf{p} \in \mathscr{R}_n$ (be a reactive-n strategy) and the game be additive. Then there exists $\tilde{\mathbf{p}}$ in the set of pure self-reactive-$(n-1)$ strategies that is a best response.

n	1	2	3	4	5
pure memory-n	16	65,536	1,844,674,4 ✗ 70,955,161		
pure self-reactive-n	4	16	256	65,536	
pure self-reactive-$(n-1)$	2	4	16	256	

Number of equations to be checked.

Theorem. Let $\mathbf{p} \in \mathscr{R}_n$ (be a reactive-n strategy) and the game be additive. Then there exists $\tilde{\mathbf{p}}$ in the set of pure self-reactive-$(n-1)$ strategies that is a best response.

n	1	2	3	4	5
pure memory-n	16	65,536	1,844,674,4	✗ 70,95	✗ 1
pure self-reactive-n	4	16	256	65,536	4,294,967,296
pure self-reactive-$(n-1)$	2	4	16	256	65,536

Number of equations to be checked.

1. Introduction and motivation

2. Conditional cooperation with longer memory

Published *PNAS*: https://doi.org/10.1073/pnas.2420125121

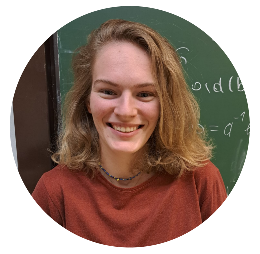


3. Complete strategy spaces of direct reciprocity | Can I afford to remember less than you?

Under review
PNAS

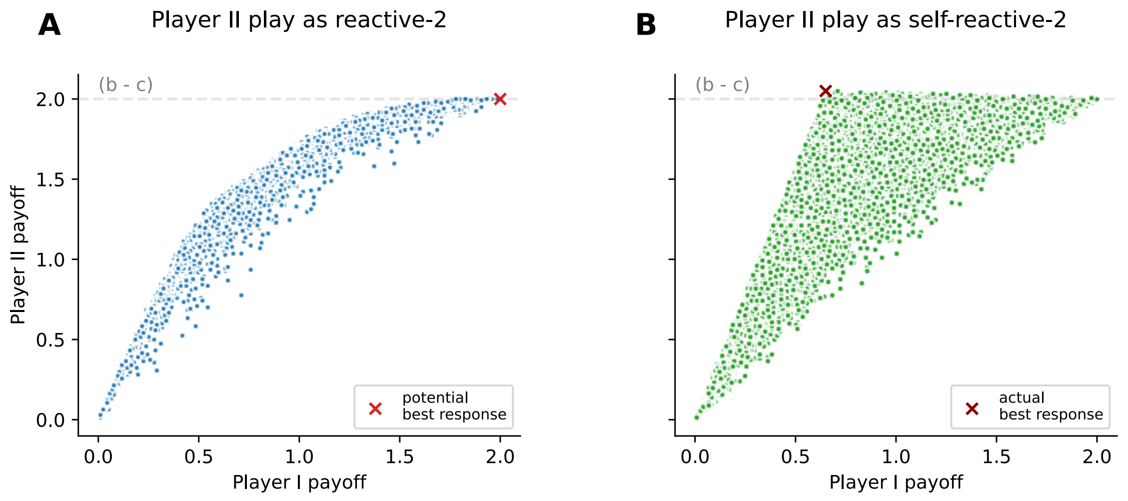


Under review
Economics Letters



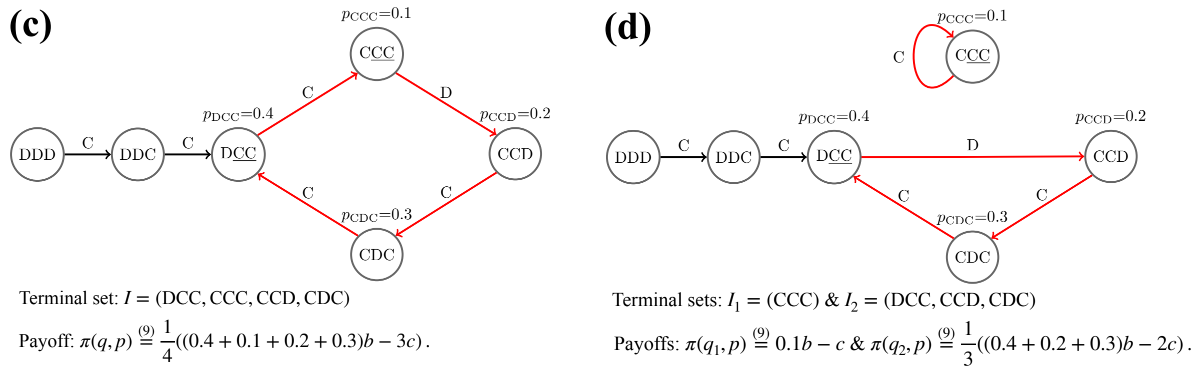

1.

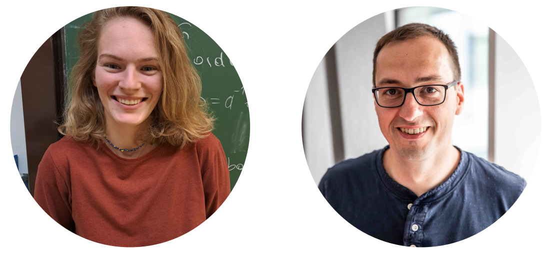
Complete strategy spaces in memory-n strategies.



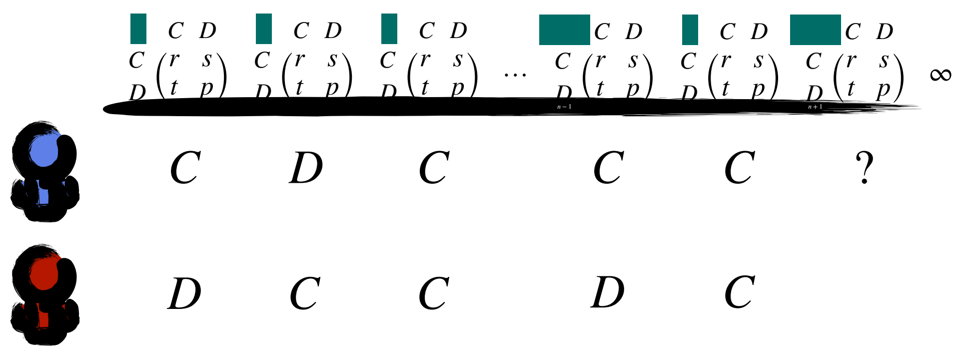
2.

There exists a best response in pure self-reactive $n-1$ for additive games for any actions for non symmetric games





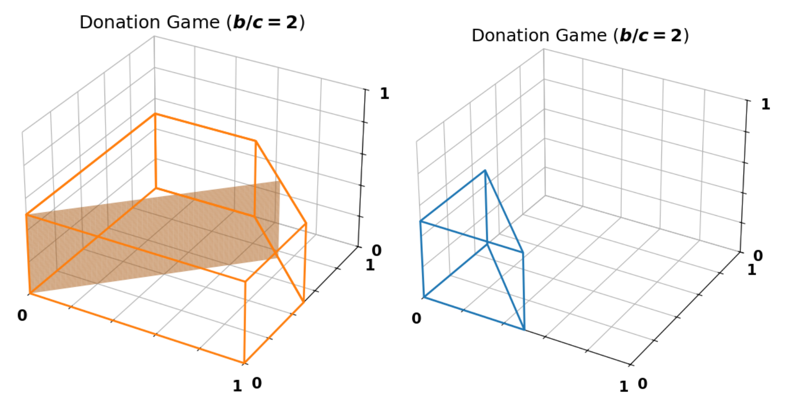
Best Responses



Nash Equilibria

n	1	2	3	4	5
pure memory-n	16	65,536	1,844,674,4	✗ 70,955	✗
pure self-reactive-n	4	16	256	65,536	4,294,967,296
pure self-reactive-$(n-1)$	2	4	16	256	65,536

Evolution of Cooperation



Conditional cooperation with longer memory: https://doi.org/10.1073/pnas.242012512

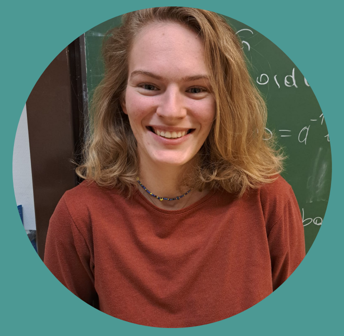
@nikoletaglyn.bsky.social

Github: Nikoleta-v3

https://nikoleta-v3.github.io/





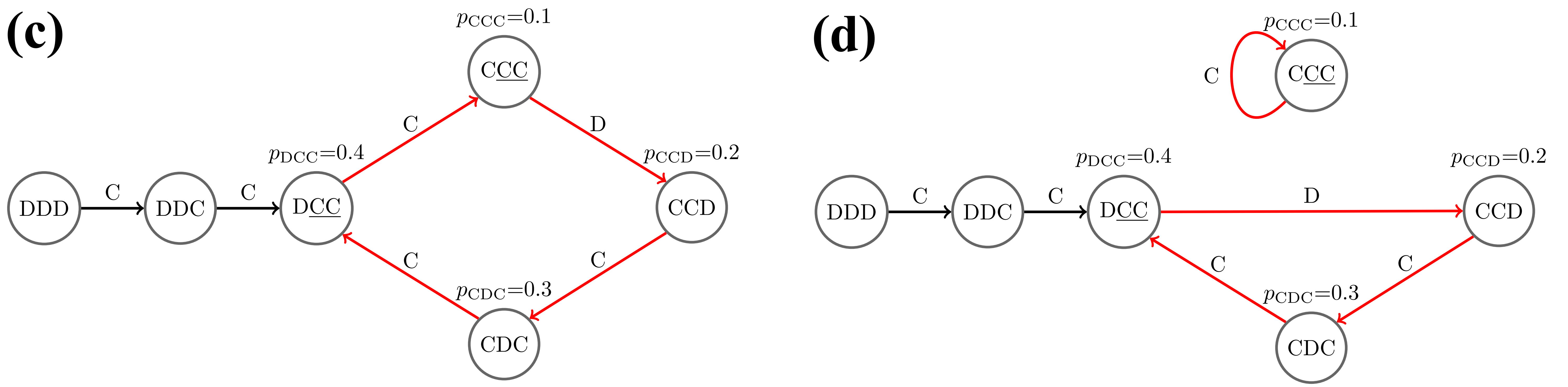


philiplaporte.github.io/

franzilesi.bsky.social

Thank you!

MAX-PLANCK-GESELLSCHAFT RIKEN R-CCS



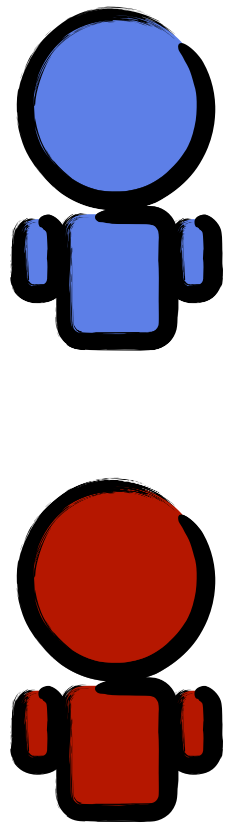
Terminal set: $I = (\mathrm{DCC}, \mathrm{CCC}, \mathrm{CCD}, \mathrm{CDC})$

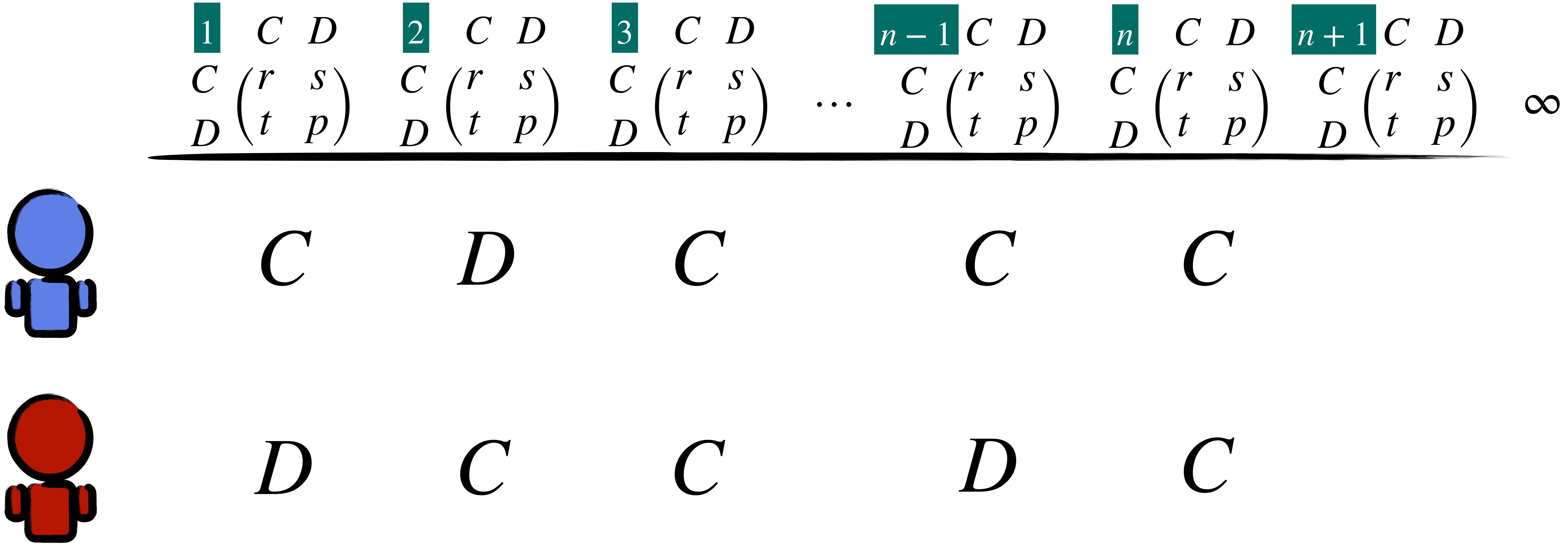
Payoff: $\pi(q,p) \overset{(9)}{=} \frac{1}{4}((0.4 + 0.1 + 0.2 + 0.3)b - 3c)$.

Terminal sets: $I_1 = (\mathrm{CCC})$ & $I_2 = (\mathrm{DCC}, \mathrm{CCD}, \mathrm{CDC})$

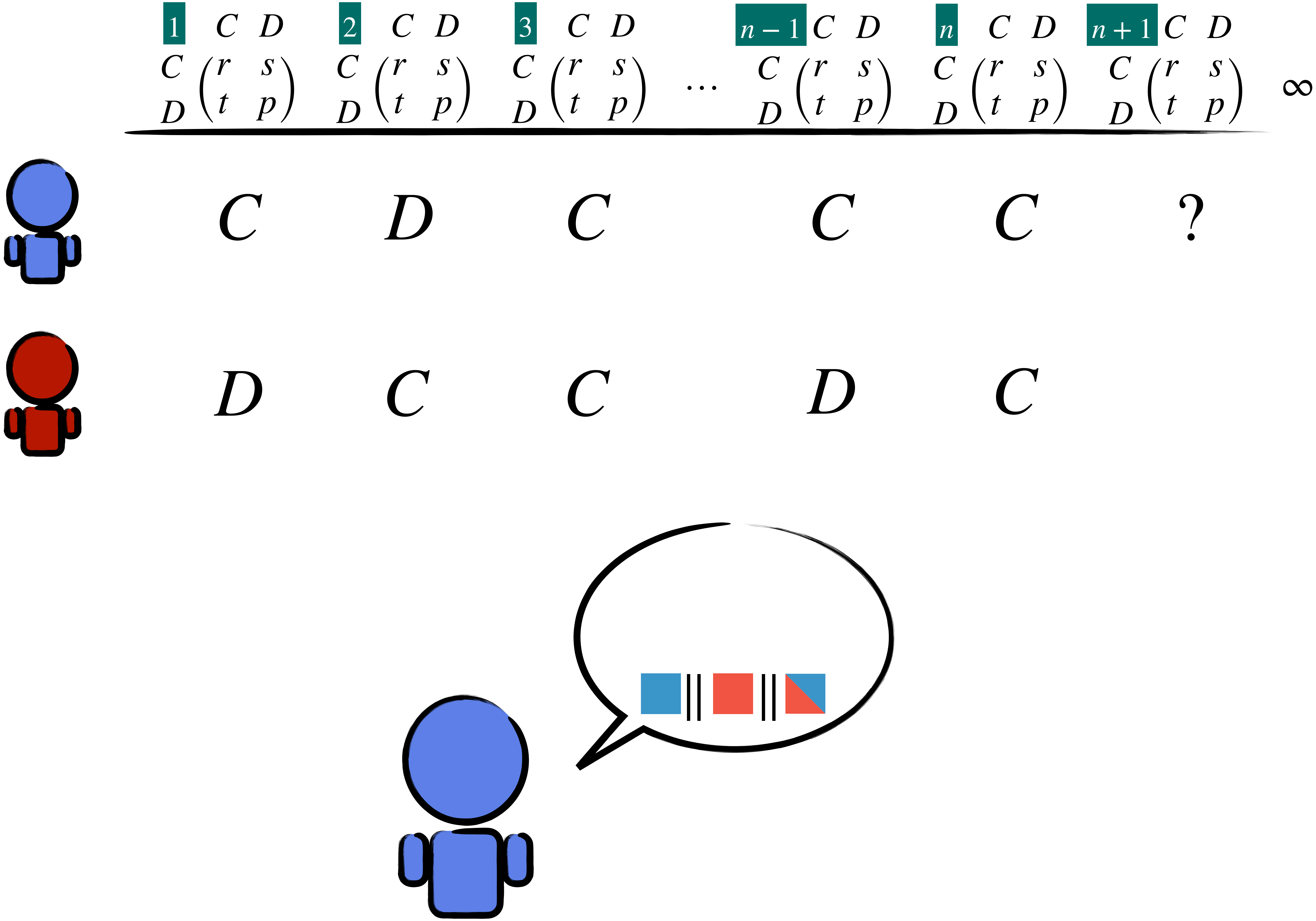
Payoffs: $\pi(q_1,p) \overset{(9)}{=} 0.1b - c$ & $\pi(q_2,p) \overset{(9)}{=} \frac{1}{3}((0.4 + 0.2 + 0.3)b - 2c)$.

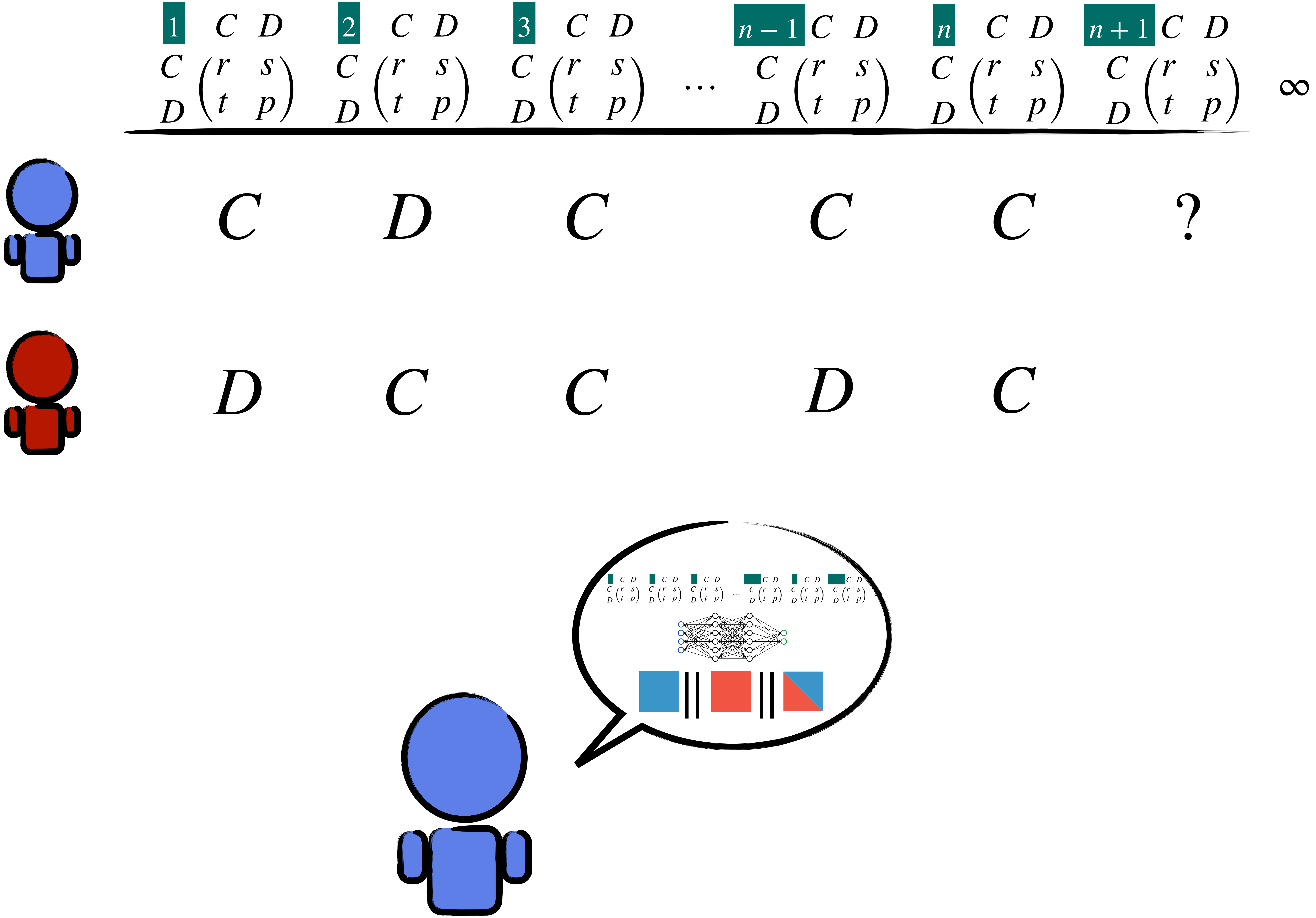
$$\boxed{1}\quad \begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} \qquad \boxed{2}\quad \begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} \qquad \boxed{3}\quad \begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} \qquad \cdots \qquad \boxed{n-1}\quad \begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} \qquad \boxed{n}\quad \begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} \qquad \boxed{n+1}\quad \begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} \qquad \infty$$

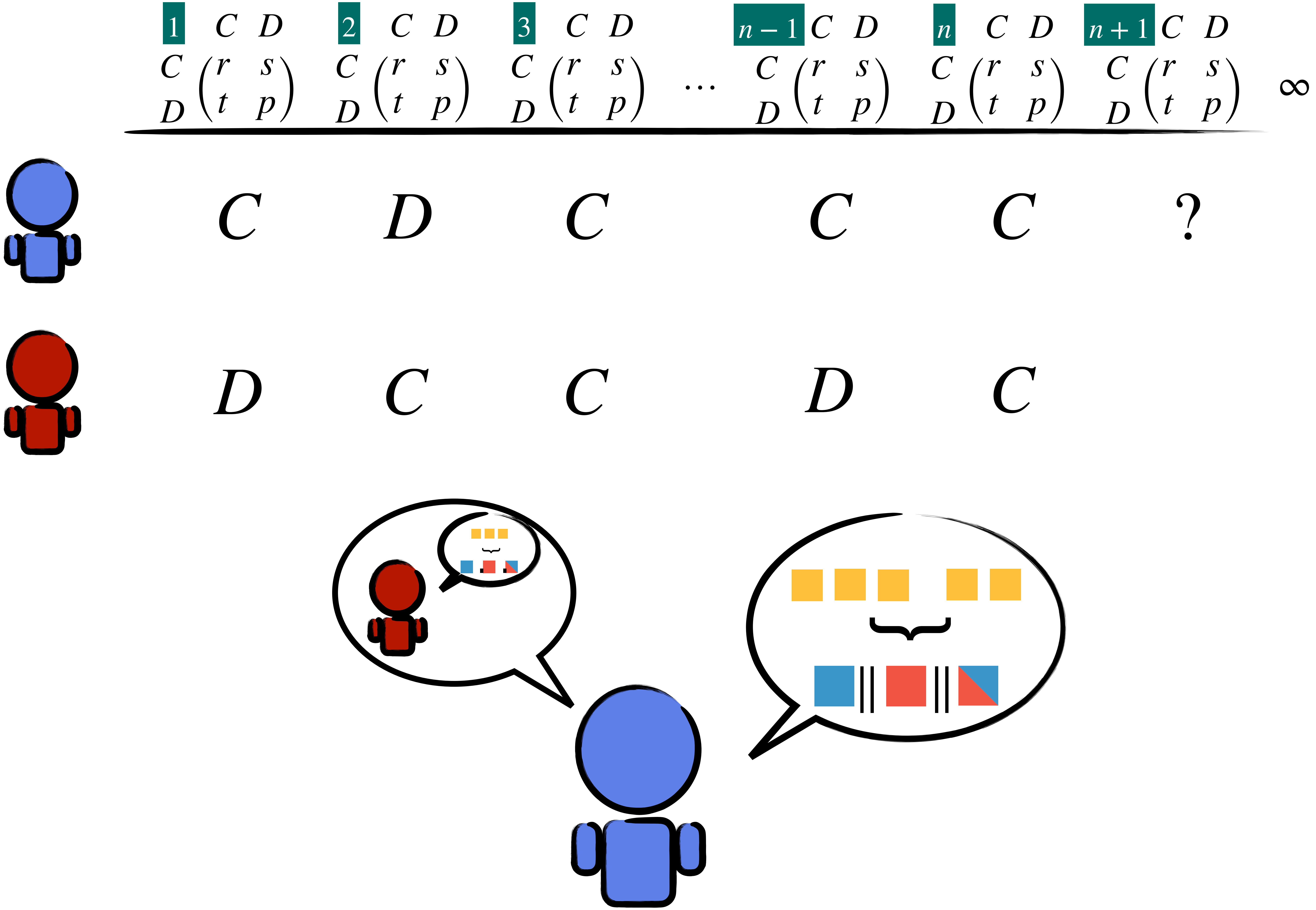




1 C D
C (r s)
D (t p)
2 C D
C (r s)
D (t p)
3 C D
C (r s)
D (t p)
…
n − 1 C D
C (r s)
D (t p)
n C D
C (r s)
D (t p)
n + 1 C D
C (r s)
D (t p)
∞
C D C C C
D C C D C


1 2 3 … n − 1 n n + 1 ∞
C D
C (r s)
D (t p)
C D C C C ?
D C C D C


1 C D C (r s) D (t p)
2 C D C (r s) D (t p)
3 C D C (r s) D (t p)
…
n − 1 C D C (r s) D (t p)
n C D C (r s) D (t p)
n + 1 C D C (r s) D (t p)
∞
C D C C C ?
D C C D C


1
2
3
n − 1
n
n + 1
C D
C r s
D t p
…
∞
C D C C C ?
D C C D C